Structures Markoviennes cachées et modèles à corrélations conditionnelles dynamiques: extensions et applications aux corrélations d'actifs financiers.

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Structures Markoviennes cachées et modèles à corrélations conditionnelles dynamiques : extensions et applications aux corrélations d'actifs financiers

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Résumé

L'objectif de cette thèse est d'étudier le problème de la modélisation des changements de régime dans les modèles à corrélations conditionnelles dynamiques en nous intéressant plus particulièrement à l'approche Markov-switching. A la différence de l'approche standard basée sur le modèle à chaîne de Markov caché (HMM) de base, nous utilisons des extensions du modèle HMM provenant des modèles graphiques probabilistes. Cette discipline a en effet proposé de nombreuses dérivations du modèle de base permettant de modéliser des structures complexes. Cette thèse se situe donc à l'interface de deux disciplines : l'économétrie financière et les modèles graphiques probabilistes.

Le premier essai présente un modèle construit à partir d'une structure hiérarchique cachée markovienne qui permet de définir différents niveaux de granularité pour les régimes. Il peut être vu comme un cas particulier du modèle RSDC de Pelletier (2006). Basé sur le HMM hiérarchique de Fine, Singer, and Tishby (1998) notre modèle permet de capter des nuances de régimes qui sont ignorées par l'approche Markov-Switching classique.

La seconde contribution propose une version Markov-switching du modèle de Engle and Sheppard (2001) construite à partir du modèle HMM factorisé de Ghahramani and Jordan (1997). Alors que l'approche Markov-switching classique suppose que les tous les éléments de la matrice de corrélation suivent la même dynamique, notre modèle permet à tous les éléments de la matrice de corrélation d'avoir leur propre dynamique de saut.

Dans la dernière contribution, nous proposons un modèle DCC construit à partir d'un arbre de décision. L'objectif de cet arbre est de relier le niveau des volatilités individuelles avec le niveau des corrélations. Pour cela, nous utilisons un arbre de décision Markovien caché, qui est une extension de HMM développée par Jordan, Ghahramani, and Saul (1997).

Mots clés : Modèle GARCH multivariés ; corrélations conditionnelles dynamiques ; modèle de Markov caché ; modèle de Markov caché hiérarchique ; modèle de Markov caché factorisé ; arbre de décision Markovien caché.

Classification JEL : C32, C51, G1, G0.

Abstract

The objective of this thesis is to study the modeling of change in regime in the dynamic conditional correlation models. We focus particularly on the Markov-switching approach. Unlike the standard approach based on the Hidden Markov Model (HMM), we use extensions of HMM coming from probabilistic graphical models theory. This discipline has in fact proposed many derivations of the basic model to model complex structures. Thus, this thesis is at the interface between two disciplines : financial econometrics and probabilistic graphical models.
The first essay presents a model constructed from a hierarchical hidden Markov which allows to increase the granularity of the regimes. It can be viewed as a special case of RSDC model of Pelletier (2006). Based on the hierarchical HMM of Fine, Singer, and Tishby (1998), our model can capture nuances of regimes than is possible with the classical Markov-Switching approach.
The second contribution proposes a Markov-switching version of the DCC model of Engle and Sheppard (2001). It is based on the factorial HMM of Ghahramani and Jordan (1997). While the classical Markov-switching approach assumes that all elements of the correlation matrix follow the same switching dynamic, our model allows all elements of the correlation matrix to have their own switching dynamic.
In the final contribution, we propose an extension of the DCC model based on a stochastic decision tree. Our model links the univariate volatilities with the correlations via a hidden stochastic decision tree. The ensuing Hidden Markov Decision Tree (HMDT) model is in fact an extension of the Hidden Markov Model (HMM) introduced by Jordan, Ghahramani, and Saul (1997). The architecture of this model is the opposite of the classical deterministic approach based on a binary decision tree and it allows a probabilistic point of view of the relationship between univariate volatility and correlations.

Keywords : Multivariate GARCH ; Dynamic conditional correlations ; Regime switching ; Hidden Markov model ; Hierarchical hidden Markov model ; Factorial Hidden Markov models ; Hidden Markov decision tree.
JEL Classification : C32, C51, G1, G0.

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CHAPITRE

1

Introduction générale.

L'objet de cette thèse est d'étudier le problème des changements de régime dans les modèles à corrélations conditionnelles dynamiques en proposant des modélisations issues des modèles graphiques probabilistes. Après avoir effectué une brève revue des modèles utilisés traditionnellement en économétrie financière pour modéliser la volatilité, nous présentons plus spécifiquement les modèles GARCH multivarié à corrélations conditionnelles dynamiques. Un bref état de l'art permettra ensuite de pouvoir cerner les avancées récentes dans ce domaine. La troisième partie de cette introduction apporte quelques éléments de base concernant les modèles graphiques probabilistes et présente succinctement quelques-uns des modèles que nous utiliserons par la suite. La dernière partie de l'introduction offre un résumé des contributions de cette thèse.

1.1 Origines des modèles à corrélations conditionnelles

1.1.1 Séries financières et faits stylisés

L'idée de représenter l'évolution d'une série financière par une loi de probabilité apparaît pour la première fois dans la thèse de Bachelier (1900). La détermination de la nature de la loi sous jacente a depuis focalisée une attention croissante des chercheurs en économie et statistique.

Les variations d'une série financière à la hausse ou à la baisse avec une amplitude plus ou moins forte caractérisent la *volatilité*. L'étude de la volatilité d'une série financière revient donc à étudier son rendement, c'est-à-dire l'étude de la trajectoire de ses variations. En notant S_t le cours d'une série financière à la date t, le rendement r_t consiste en une transformation logarithmique du cours :

$$r_t = \log S_t - \log S_{t-1} \tag{1.1}$$

$$\simeq \frac{S_t - S_{t-1}}{S_{t-1}} \tag{1.2}$$

Le rendement r_t peut ainsi être interprété comme la variation relative du cours d'une série financière. Cette transformation permet ainsi de comparer plusieurs séries.

La recherche des spécifications relatives à la distribution des rendements a été influencée par des études mettant en évidence certaines caractéristiques et régularités statistiques propres aux séries financières. Ces faits stylisés communs aux séries financières ont été et sont encore largement discutés dans la littérature. Nous exposons brièvement ces régularités statistiques :

- non stationnarité : le cours d'une série financière n'est généralement pas stationnaire au second ordre.
- absence d'autocorrélation des variations de prix : les autocorrélations du cours sont généralement très faibles. Ce fait renvoie à l'hypothèse de bruit blanc du cours.
- autocorrélation de carrés des variations de prix : les carrés des rendements présentent généralement de fortes autocorrélations.
- clustering : de fortes variations des rendements sont suivies de fortes variations. Ce regroupement des extrêmes est évoqué dans l'article précurseur de Mandelbrot (1963b) [1].
- queues de distribution épaisses : les cours ainsi que les rendements sont leptokurtiques. Cette caractéristique, qui tend à rejeter le paradigme gaussien, a été notamment popularisée par les articles de Mandelbrot (1963a) et Fama (1965).

1. "Large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes."

– effets de levier : cet effet fait référence à la réponse asymétrique des valeurs passées positives et négatives sur la volatilité. Concrètement, les valeurs négatives ont tendance à provoquer une augmentation de la volatilité supérieure à celle induite par une hausse du cours d'une ampleur similaire. Black (1976) montre que l'effet de levier a tendance à s'accompagner d'un effet d'asymétrie alors que la réciproque n'est pas vérifiée.
– saisonnalité : la volatilité tend à augmenter lorsque les marchés ne fonctionnent pas. De nombreux effets de saisonnalités apparaissent en fonction de la fréquence des observations. Parmi ceux-ci, citons le cas de l'effet week-end (voir French (1980)) et l'effet janvier (voir Keim (1983)).

Le lecteur intéressé par un exposé plus complet de ces faits stylisés pourra consulter le premier chapitre de la monographie de Francq and Zakoian (2009) ainsi que l'article de Cont (2001).

Les méthodes classiques d'analyse des séries temporelles comme les modèles de type ARMA ne sont globalement pas adaptées pour reproduire de telles propriétés statistiques dans la mesure où une formulation simple de type ARMA ne permet pas de traduire une relation non linéaire. Rappelons qu'un processus ARMA(p,q) est de la forme :

$$X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = \varepsilon_t + \sum_{j=1}^{q} \theta_t \varepsilon_{t-1} \forall t \text{ avec } \phi_i, \theta_i \in \mathbb{R} \text{ et } \in_t \rightsquigarrow WN(0,\sigma) \tag{1.3}$$

La valeur présente des séries est écrite comme une fonction linéaire des valeurs passées et de la valeur présente d'un bruit, qui est l'innovation de la série. Cette formulation présente l'intérêt d'être simple à mettre en œuvre, mais elle est cependant beaucoup trop restrictive. La linéarité des modèles ARMA s'exprime avec le théorème de Wold. C'est-à-dire que tout processus centré et stationnaire peut être représenté sous une forme MA infini :

$$X_t = \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} + \eta_t \text{ avec } \eta_t \rightsquigarrow WN(0,\sigma) \text{ et} \sum_j |\theta_j| < \infty \tag{1.4}$$

Autrement dit, tout processus faiblement stationnaire peut être reformulé sous la forme d'une combinaison linéaire d'une suite de variables aléatoires non corrélées dans le temps. Cette hypothèse forte de linéarité ne permet donc pas de prendre en compte des comportements d'asymétrie, de rupture ou l'existence de cluster de volatilité.

1.1.2 Modélisation de la volatilité

Pour tenter de reproduire les propriétés statistiques évoquées précédemment, il a donc fallu s'orienter vers des spécifications non-linéaires. Dans ce cadre, les modèles économétriques sont généralement construits à partir de la représentation multiplicative suivante :

$$r_t = h_t^{1/2} \eta_t \tag{1.5}$$

où η_t est une suite de variables aléatoires indépendantes et identiquement distribuées selon une loi de probabilité discrète, de moyenne nulle et de variance unité. Le terme h_t représente la variance conditionnelle du processus r_t. Dans cette formulation, il est donc possible de construire un processus permettant de reproduire certains des faits stylisés en jouant sur le choix de distribution de η_t mais aussi en spécifiant la formulation de h_t. Plusieurs classes de modèles ont émergé dans la littérature quant à la spécification de la variance conditionnelle. Nous en présentons les principales.

Modèles à volatilité conditionnellement hétéroscédastique Introduits par Engle (1982), les modèles de type ARCH (Auto Regressive Conditional Heteroskedasticity) sont construits à partir de l'hypothèse que la variance des perturbations évolue avec le temps :

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 \tag{1.6}$$

Ainsi, dans cette configuration le carré des perturbations suit un processus autorégressif d'ordre p. Ce premier modèle à variance conditionnellement hétéroscédastique a été rapidement généralisé par Bollerslev (1986) en introduisant une composante moyenne mobile pour aboutir au modèle GARCH (Generalized Auto Regressive Conditional Heteroskedasticity) :

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i r_{t-i}^2 + \sum_{i=1}^{q} \beta_i h_{t-i} \tag{1.7}$$

Les paramètres à estimer du modèles sont alors $\{\omega, \alpha_i, \beta_j\}$, $i = 1, ..., p$ et $j = 1, ..., q$. Cette classe de modèle a rapidement été très populaire en raison de sa facilité à être estimée.

Modèles à volatilité stochastique Une autre classe de modèle est celle composée des modèles à volatilité stochastique. Cette approche proposée par Taylor (1982) est définie comme :

$$\log h_t = \omega + \phi \log h_{t-1} + v_t \tag{1.8}$$

où les innovations v_t sont indépendantes de η_t et avec $v_t \overset{iid}{\sim} \mathcal{N}(0, \sigma)$. Les paramètres à estimer sont $\{\omega, \eta, \phi\}$. Cette approche possède donc deux sources d'aléa : v_t correspond à l'impact du flux d'information tandis que η_t représente la nature de l'information (bonne ou mauvaise). A la différence des modèles GARCH, où la variance est spécifiée directement à partir des valeurs observées des rendements, les modèles à volatilité stochastique spécifient la volatilité comme un processus latent. Cette particularité rend l'estimation de ces modèles plus compliquée que la classe précédente.

Modèles à volatilité multifractale Cette approche trouve ses origines dans les travaux de Mandelbrot (1967, 1974, 1982, 1988). Un premier modèle, nommé MMAR (Multifractal Model of Asset Returns) basée sur cette approche prenant en compte des relations d'échelle a été proposée par Calvet, Fisher, and Mandelbrot (1997). Ce modèle a ensuite été enrichi pour aboutir au modèle MSM (Markov-Switching Multifractal) par Calvet and Fisher (2001, 2002) où la volatilité est déterminée par le produit de $\bar{k}$ composantes aléatoires $M_{k,t}$:

$$h_t = \bar{\sigma} \left(\prod_{i=1}^{\bar{k}} M_{k,t} \right)^{1/2} \tag{1.9}$$

où $\bar{\sigma}$ est un paramètre à estimer. A chaque instant les composantes de volatilités suivent une dynamique markovienne au premier ordre avec une probabilité de transition spécifiée comme :

$$\gamma_k = 1 - (1 - \gamma_1)^{(b^{k-1})} \tag{1.10}$$

avec b et γ_k des paramètres devant être estimés.

Parce qu'ils ne reproduisent que partiellement les régularités statistiques évoquées précédemment, ces modèles pionniers ont connu de nombreux raffinements. En particulier de nombreuses spécifications non linéaires ont été proposées. Cette sophistication des modèles de base est à l'origine

d'une vaste littérature que nous ne pouvons résumer dans cette thèse [2]. Par ailleurs, ces différents axes de recherches illustrent la difficulté de modélisation de la volatilité des marchés financiers. Les origines des régularités statistiques observées sur les séries financières ne sont pas encore parfaitement comprises aujourd'hui. Leur modélisation statistique pose des problèmes théoriques complexes. A ce jour, aucune des approches présentées ne s'est réellement imposée et l'économétrie financière reste un domaine en constante évolution.

1.2 Cadre général des modèles à corrélations dynamiques.

L'extension des modèles GARCH univariés au cadre multivarié est motivée par l'intérêt de décrire la volatilité de plusieurs séries temporelles conjointement, afin d'exploiter autant que possible les liens qui peuvent exister entre ces séries [3]. Dans ce cadre, la volatilité conditionnelle d'une série vectorielle de rendements r_t, de dimension $K \times 1$ s'écrit à partir de la formulation :

$$r_t = H_t^{1/2} \eta_t \tag{1.11}$$

où $H_t = [h_{ijt}]$ est la matrice de covariance des rendements de taille $K \times K$ et η_t un processus de dimension indépendant et identiquement distribué de moyenne nulle et tel que $\mathbb{E}(\eta_t \eta_t) = \mathbb{I}$ où $\mathbb{I}$ est une matrice identité de taille K.

1.2.1 Origines

L'extension des modèles GARCH au cadre multivarié n'est pas immédiate et fait apparaître plusieurs difficultés. Une généralisation directe du modèle univarié entraînerait une inflation du nombre de paramètres à estimer conduisant très rapidement à des problèmes numériques. Les développements de modèles GARCH multivariés doivent donc concilier une spécification plus ou moins parcimonieuse en paramètres tout en préservant une certaine flexibilité pour ne pas perdre de leur pouvoir explicatif. Deux classes de modèles ordonnent ces extensions. Dans la première, la matrice des covariances conditionnelles est modélisée directement. Dans la seconde, cette matrice est obtenue en intégrant les corrélations conditionnelles.

Modélisation directe de H_t Dans cette première classe de modèles, la matrice de covariance conditionnelle est modélisée directement et adopte donc la formulation :

$$H_t = f(H_{t-1}, H_{t-2}, ..., r_{t-1}, r_{t-2}, ...) \tag{1.12}$$

où f est une fonction de transformation mesurable par rapport à l'ensemble d'informations à la date $t-1$, noté $\mathscr{F}_{t-1}$. Une des premières contributions est celle de Bollerslev, Engle, and Wooldridge (1988) avec le modèle VEC. Ce modèle est une généralisation du modèle de base univarié. Cette première formulation, si elle présente un intérêt au niveau théorique n'est pas satisfaisante

2. L'ouvrage de Engle (1995) regroupe les articles fondateurs des modèles GARCH. La synthèse de Silvennoinen and Teräsvirta (2009) présente un panorama des modèles GARCH univariés. On pourra également se référer au travail de Bollerslev (2008) qui recense de façon exhaustive les différentes déclinaisons ainsi qu'à la monographie de Francq and Zakoian (2009) qui présente un état récent de la recherche théorique de cette classe de modèle. Les fondements des modèles à volatilité stochastiques sont présentés dans le recueil d'articles de Shephard (2005) ainsi que dans l'article de Shephard and Andersen (2009). Les modèles à volatilité multifractale sont l'objet d'un ouvrage de synthèse de Calvet and Fisher (2008).

3. D'un point de vue concret, une formulation multivariée peut alors apporter une information importante afin de construire un portefeuille composé d'actifs très variés. Ainsi, les covariances entre les actifs constituent un outil de décision crucial dans le CAPM.

car trop générale. Une formulation contrainte est également proposée avec le modèle diagonal-VECH, impliquant que les éléments de la matrice H_t dépendent du carré des résidus passés et d'un terme autorégressif. Les covariances sont alors une combinaison linéaire du produit croisé des résidus passés ainsi que d'un terme autorégressif. Cependant, cette configuration, même contrainte demeure très lourde au niveau numérique. Dans la continuité du modèle VEC, Engle and Kroner (1995) proposent la représentation BEKK [4], imposant des contraintes moins fortes sur les paramètres et permettant des conditions plus faibles pour respecter la stationnarité du processus. Par ailleurs, l'avantage considérable de cette formulation réside dans le fait que la matrice H_t est écrite sous une forme quadratique. Il en résulte qu'elle est donc toujours définie positive, allégeant par là même les difficultés numériques de l'estimation. Une solution alternative à l'écriture quadratique est proposée par Kawakatsu (2003), qui consiste à construire la matrice de covariance à partir d'une représentation de Choleski.
Pour limiter l'inflation de paramètre, plusieurs chercheurs ont proposé des modèles basés sur des facteurs. Ainsi, Engle, Ng, and Rothschild (1990) exploitent l'idée que les éléments dépendant du temps dans la matrice de variance-covariance sont définis par une fonction de combinaisons linéaires de variables aléatoires. L'approche factorielle est également exploitée par Alexander (2002), van der Weide (2002) ainsi que Vrontos, Dellaportas, and Politis (2003) qui proposent un modèle GARCH multivarié basé sur l'analyse en composantes principales [5]. Lanne and Saikkonen (2007) réalisent une spécification autorisant un nombre réduit de facteurs conditionnellement hétéroscédastiques (modèle GOF GARCH), contrairement aux approches orthogonales précédentes. Enfin, le modèle présenté par García-Ferrer, González-Prieto, and Peña (2008) peut être considéré comme une raffinement du GOF-GARCH dans lequel la sélection des facteurs est réalisée par une analyse en composantes indépendantes au lieu d'une analyse en composantes principales (modèle GICA GARCH).
Des spécifications semi/non paramétriques ont également été proposées. Hafner and Rombouts (2007) établissent un modèle semi-paramétrique dans lequel la matrice de covariance est modélisée par une spécification paramétrique (comme un modèle VEC ou BEKK) tandis que l'erreur de distribution est estimée non paramétriquement. Une approche similaire, combinant une composante paramétrique et une composante non paramétrique, est proposée par Long and Ullah (2005).

Modèles à corrélations conditionnelles Les modèles à corrélations conditionnelles diffèrent de la formulation explicitée dans le paragraphe dans le sens où la matrice de covariance conditionnelle incorpore une matrice de corrélation conditionnelle. Initiée par Bollerslev (1990) avec le modèle à corrélations conditionnelles constantes (CCC), cette classe repose sur la décomposition suivante :

$$H_t = D_t R D_t \tag{1.13}$$

où R est, dans l'article pionnier de Bollerslev (1990), une matrice de corrélation conditionnelle constante dans le temps de taille $K \times K$. Le terme D_t est une matrice diagonale de taille $K \times K$ dont les éléments contiennent les écarts types conditionnels de chaque série :

$$D_t = \text{diag}\{h_{i,t}^{1/2}\},\ i = 1, ..., K \tag{1.14}$$

Cette information est extraite grâce à un modèle univarié à variance conditionnelle de la famille GARCH.
La représentation définie par l'équation (1.13) implique donc une modélisation de la matrice de covariance conditionnelle en deux étapes. Dans un premier temps, l'information concernant la volatilité individuelle est extraite avec une filtration de type GARCH. Cette filtration [6] permet

4. Le nom de cette formulation provient d'une synthèse des analyses réalisées concernant les représentations multivariées par Robert F. Engle, Kenneth F. Kroner, Yoshihisa Baba et Dennis F. Kraft.
5. Modèles Orthtogonal GARCH, Generalized Orthogonal GARCH et Full Factor GARCH.
6. L'étape d'extraction de la volatilité individuelle est appélée par Engle (2009) *degarching*.

d'obtenir les résidus standardisés de la série vectorielle de rendements :

$$\varepsilon_t = D_t^{-1} r_t \tag{1.15}$$

Les corrélations conditionnelles correspondent alors à l'espérance des résidus standardisés :

$$\mathbb{E}_{t-1}[\varepsilon_t \varepsilon_t'] = D_t^{-1} H_t D_t^{-1} = R \tag{1.16}$$

avec R une matrice carrée symétrique définie positive. La matrice de covariance conditionnelle est ainsi une combinaison des variances conditionnelles individuelles et de la variance conditionnelle des résidus standardisés. D'un point de vue pratique, cette formulation présente un avantage numérique indéniable. En effet, la première étape relative à la filtration GARCH des séries individuelles correspond à une somme de problèmes numériques faciles à traiter tandis que l'estimation de la variance des résidus standardisés peut s'effectuer dans un second temps. Cette estimation en plusieurs étapes décompose donc un problème numérique complexe en une somme de problèmes plus faciles à traiter.
Néanmoins, si le modèle CCC constitue une avancée théorique importante, il apparaît comme trop restrictif pour une utilisation concrète. L'hypothèse de corrélations constantes sur les marchés financiers n'est en effet pas vérifiée empiriquement [7]. Les recherches se sont alors focalisées sur l'introduction de corrélations variant dans le temps.

1.2.2 Modèles à corrélations dynamiques conditionnelles

L'introduction de corrélations conditionnelles dynamiques consiste à donner un caractère temporel à la matrice de corrélation, l'équation (1.13) étant alors reformulée comme :

$$H_t = D_t R_t D_t \tag{1.17}$$

Deux spécifications dynamiques ont inauguré presque simultanément cette extension du modèle de base. Tse and Tsui (2002) proposent ainsi d'écrire les corrélations conditionnelles à partir d'un processus autorégressif (modèle DCC_T) :

$$R_t = (1-a-b)\overline{R} + aR_{t-1} + b\Psi_{t-1} \tag{1.18}$$

où $\overline{R}$ est la matrice des corrélations non conditionnelles, Ψ_{t-1} une matrice de corrélation empirique des résidus standardisés et $a+b \leq 1$. Ce modèle est donc basé sur l'hypothèse que les corrélations sont une fonction des corrélations passées ainsi que des corrélations empiriques.
Parallèlement à Tse and Tsui (2002), Engle and Sheppard (2001) proposent également une écriture autorégressive des corrélations conditionnelles dynamiques (modèle DCC_{ES}). Cependant, leur approche diffère de la précédente car les corrélations ne sont pas formulées comme une fonction des corrélations retardées. Ainsi, au lieu de modéliser directement les corrélations, Engle and Sheppard (2001) imposent une dynamique de type BEKK au processus des covariances des résidus standardisés :

$$Q_t = \Omega + a\varepsilon_{t-1}\varepsilon_{t-1}' + bQ_{t-1} \tag{1.19}$$

avec a et b deux paramètres vérifiant $a+b<1$. Les corrélations sont alors obtenues pour une combinaison des covariances :

$$R_t = \text{diag}\{Q_t\}^{-1/2} Q_t \text{diag}\{Q_t\}^{-1/2} \tag{1.20}$$

7. L'hypothèse de corrélations constante est d'abord rejetée dans une analyse de Longin and Solnik (1995). L'étude de Longin and Solnik (2001) montre que les corrélations ont tendance à augmenter davantage dans une situation de forte volatilité à la baisse que lors d'une forte volatilité à la hausse. Ce caractère asymétrique du mouvement des corrélations est confirmé dans les articles de Ang and Chen (2002) et Campbell, Koedijk, and Kofman (2002).

La constante ω est estimée indépendamment par *variance targeting*. Ainsi, les auteurs contournent le problème délicat de l'estimation de la constante en ayant uniquement recours à un estimateur des moments. Cette constante est alors estimée par :

$$\hat{\Omega} = (1 - a - b)\overline{Q} \tag{1.21}$$

avec

$$\overline{Q} = \frac{1}{T}\sum_{t=1}^{T} \varepsilon_t \varepsilon_t \tag{1.22}$$

Dans la formulation de Engle and Sheppard (2001), le processus suivi par les corrélations ne nécéssite que deux paramètres. Les modèles de type DCC ont rapidement connu un grand succès. Outre le fait d'incorporer les corrélations dans l'écriture des covariances conditionnelles, c'est leur facilité d'estimation qui rend ce modèle attractif pour le praticien [8].
Néanmoins, cette spécification parcimonieuse en paramètres impose une dynamique des corrélations très restrictive. Diverses extensions ont alors été proposées afin d'apporter davantage de flexibilité et enrichir le processus suivi par les corrélations. Nous allons tenter d'en exposer les principales.

Ecritures vectorielle ou matricielles contraintes. L'écriture scalaire de l'équation (1.19) étant très réductrice, Engle (2002) propose une écriture matricielle permettant d'avoir des coéfficients de covariations spécifiques (modèle G-DCC). L'équation (1.19) se reformule alors comme :

$$Q_t = [\overline{Q} - A'\overline{Q}A - B'\overline{Q}B] + A'\varepsilon_{t-1}\varepsilon'_{t-1}A + B'Q_{t-1}B \tag{1.23}$$

dans laquelle A et B sont deux matrices de taille $K \times K$. Cette formulation très générale impose cependant des conditions de stationnarité délicates à respecter puisque que $\overline{Q} - A'\overline{Q}A - B'\overline{Q}B$ doit être une matrice semi-définie. Dans le but de réduire le nombre de paramètres, une écriture vectorisée peut être une solution efficace. En remplaçant les matrice A et B par des vecteurs l'équation (1.23) s'écrit :

$$Q_t = [\iota\iota' - A - B] \circ \overline{Q} + A \circ \varepsilon_{t-1}\varepsilon'_{t-1} + B \circ Q_{t-1} \tag{1.24}$$

Néanmoins, si cette formulation diminue effectivement le nombre de paramètres tout en préservant une certaine généralité, les conditions pour que la constante soit semi-définie positive restent difficiles à satisfaire. Hafner and Franses (2009) proposent une version parcimonieuse de l'équation précédente en écrivant :

$$Q_t = [1 - \sum_{i=1}^{n} \alpha_i - \beta]\overline{V} + \alpha\alpha' \circ \varepsilon_{t-1}\varepsilon'_{t-1} + \beta Q_{t-1} \tag{1.25}$$

où α un vecteur de longueur K et β un scalaire.
Une autre solution pour économiser des paramètres consiste à attribuer la même dynamique aux séries similaires. Billio, Caporin, and Gobbo (2006) proposent un DCC vectorisé :

$$Q_t = cc' + \alpha\alpha' \circ \varepsilon_{t-1}\varepsilon'_{t-1} + \beta Q_{t-1} \tag{1.26}$$

où a, b, et c sont des vecteurs partitionnés. Ce modèle est généralisé par Billio and Caporin (2009) en substituant aux vecteurs partionnés des matrices par blocs.

8. En effet, ce modèle s'estime en plusieurs étapes, là où les modèles à covariances conditionelles s'estimaient en une étape impliquant un problème numérique complexe. Dans le modèle Engle and Sheppard (2001), l'estimation peut se décomposer en trois étapes : (i) *degarching*, i.e. extraction des volatilités individuelles avec un processus de type GARCH, (ii) estimation de la constante Ω par *variance targeting*, (iii) estimation des paramètres a et b avec la méthode du maximum de vraisemblance. Une explication plus approfondie de l'estimation de ce modèle se trouve dans les chapitres suivants de cette thèse.

Simplifications condensant l'information Afin de limiter le nombre de paramètres tout en gardant une certaine généralité dans la dynamique des corrélations, une possibilité consiste à regrouper les séries qui ont les mêmes dynamiques en groupe. Cette spécification avec des vecteurs partitionnés ou des matrices par blocs a été explorée notamment par Billio, Caporin, and Gobbo (2006) et Billio and Caporin (2009). Néanmoins, dans ces deux approches, l'allocation des vecteurs/matrices s'effectue manuellement, au choix du modélisateur. Un axe de recherche récent de la littérature concerne le partitionnement de la matrice de corrélation selon des techniques de *clustering*. Zhou and Chan (2008) comparent ainsi différentes méthodes de partitionnement et concluent à l'efficacité du clustering hiérarchique (modèle C-DCC1). So and Yip (2009) proposent également une approche par clusters (modèle C-DCC2) reposant sur une approche Bayésienne.
Cependant, les méthodes de partitionnement de type clusters présentent l'inconvénient d'être difficilement interprétables. Une approche alternative, suggérée par Caporin and Paruolo (2009) propose d'utiliser des méthodes de partitionnement issues de l'économétrie spatiale. Dans ce cadre, les paramètres du modèle définie par l'équation (1.23) sont déterminés à partir d'une matrice de proximité indiquant l'intensité de la dépendance d'une série par rapport à une autre. Cette approche plug-in présente l'avantage d'être interprétable tout en relâchant l'hypothèse d'un partitionnement a priori.

Prise en compte de l'asymétrie La prise en compte des effets d'asymétrie consiste à introduire un paramètre attribuant un poids aux résidus négatifs. Une extension asymétrique de l'équation (1.23) est formulée par Cappiello, Engle, and Sheppard (2006) :

$$Q_t = (\bar{V} - A'\bar{V}A - B'\bar{V}B - G'\bar{N}G) + A'\varepsilon_{t-1}\varepsilon'_{t-1}A + B'Q_{t-1}B + G'n_{t-1}n'_{t-1}G \tag{1.27}$$

avec A, B et G des matrices diagonales. L'effet d'asymétrie est capté par la variable n_t, qui vérifie : $n_t = \mathbb{1}_{\{\varepsilon_t<0\}} \circ \varepsilon_t$ et permet de calculer $\bar{N} = T^{-1}\sum_{t=1}^{T} n_t n'_t$. Les conditions de stationnarité de ce modèle impliquent que Q_0 définie positive et que la constante $(\bar{V} - A'\bar{V}A - B'\bar{V}B - G'\bar{N}G)$ soit semi définie positive.
Exploitant la même idée, Vargas (2006) suggère une extension asymétrique du modèle Block-DCC de Billio, Caporin, and Gobbo (2003). Une version asymétrique du DCC basée sur la loi multivariée de Laplace asymétrique est réalisée par Cajigas and Urga (2006).

Semi/non paramétrique DCC Une approche semi-paramétrique (modèle SPCC) est présentée par Franses, Hafner, and van Dijk (2005). C'est la matrice de covariances des résidus qui est calculée de façon non-paramétrique :

$$Q_t(x) = \frac{\sum_{t=1}^{T} \varepsilon_t \varepsilon'_t K_h(x_t - x)}{\sum_{t=1}^{T} K_h(x_t - x)} \tag{1.28}$$

où x_t est une variable de conditionnement observable (par exp. : $x_t = t$), $K_h(\cdot) = (1/h)K(\cdot/h)$ un noyau quelconque et h la fenêtre. Les corrélations sont donc calculées à partir d'un estimateur du type Nadaraya-Watson transformé.
Feng (2007) développe une estimation non paramétrique des corrélations à partir d'une méthode k-NN (k-nearest-neighbours) avec fenêtre variable (modèle local-DCC). L'idée des méthodes k-NN consiste à classer chaque observation selon une règle de décision afin de trouver les k plus proches observations.

Changements de régimes Plusieurs contributions se focalisent sur l'introduction de régimes dans le processus des corrélations. Diverses extensions du modèles à corrélations conditionnelles sont

donc apparues, basées aussi bien sur des spécifications déterministes (modèles à seuils, à transition douce) que stochastiques (modèle Markov-switching). Ces extensions sont en partie motivées par l'analyse de séries financières présentant des ruptures structurelles comme dans le cas d'une crise. Elles permettent ainsi d'isoler la dynamique correspondant à un état de crise de celui d'un état de non crise, en spécifiant un mécanisme de passage d'un état à un autre.
Reprenant l'approche de Tong and Lim (1980), Kwan, Li, and Ng (2010) proposent une spécification à seuil du modèle original de Tse and Tsui (2002) dans laquelle la variable de seuil est supposée connue (modèle VC-MTGARCH). Par ailleurs, la spécification de seuil ne concerne pas seulement les corrélations, mais aussi les volatilités univariées :

$$r_{i,t} = \Phi^j_{0,i} + \Phi^j_{1,i} r_{i,t-1} + e_{i,t} \tag{1.29}$$

$$h^j_{i,t} = \omega^j_i + \alpha^j_i r^2_{i,t-1} + \beta^j_i h_{i,t-1} \tag{1.30}$$

$$R_t = (1 - a^j - b^j)\overline{R} + a^j R_{t-1} + b^j \Psi_{t-1} \tag{1.31}$$

où l'indice j fait référence au $j$ème régime et i la $i$ème série. Les conditions de stationnarité impliquent que $\alpha^j_i + \beta^j_i < 1$ et $a^j + b^j \leq 1$. Dans ce modèle, la variable de seuil est endogène ou exogène retardée. Sur ce point, So and Yip (2010) proposent une version du modèle dans laquelle la variable de seuil s'écrit comme une moyenne pondérée ou une aggrégation de variables retardées (Aggregate Threshold DCC). .
Silvennoinen and Teräsvirta (2005) proposent le modèle Smooth Transition Conditional Correlation (STCC) dans lequel les corrélations évoluent entre deux matrices R_1 et R_2 constante dans le temps :

$$R_t = (1 - G_t)R_1 + G_t R_2 \tag{1.32}$$

Le passage de l'une à l'autre est réalisé pour une fonction de transition logistique conditionnelle à une variable de transition définie par :

$$G_t = (1 + e^{-\gamma(m_t - c)})^{-1} \tag{1.33}$$

où, γ le paramètre de lissage, c le paramètre de localisation et m_t la variable de transition. Cette approche de STAR a été par la suite enrichie par Silvennoinen and Teräsvirta (2009) avec l'introduction d'une seconde fonction de transition conditionnelle pour aboutir au Double STCC (DSTCC).
Si l'approche (S)TAR spécifie explicitement et de façon purement déterministe la fonction permettant de catégoriser les régimes, il est également possible de relâcher cette hypothèse en supposant que le mécanisme de passage d'un régime à un autre est de nature probabiliste. Diverses extensions basées sur un processus stochastique de changement de régime ont été formulées. Elles peuvent s'écrire comme :

$$R_t = \sum_{i=1}^{N} \mathbb{1}_{\{s_t = i\}} R_i \tag{1.34}$$

où R_i est le processus des corrélations correspondant au régime $i = 1, ..., N$. Une première possibilité pour modéliser un changement de régime de nature probabiliste consiste à supposer que les corrélations peuvent être réparties aléatoirement et indépendamment dans des composantes caractérisées par un régime spécifique. La variable indicatrice $\mathbb{1}_{\{s_t = i\}}$ est une variable latente qui permet de d'attribuer une probabilité non conditionnelle de répartition des corrélations dans les différentes classes spécifiques. Putintseva (2010) propose un modèle de mélanges finis en retenant l'équation (1.23) comme spécification pour les corrélations (modèle Mixture DCC 1). Ce type de modélisation est également retenu pour le modèle de Galeano and Ausín (2009), avec comme spécification celle de Tse and Tsui (2002) (modèle Mixture DCC 2).

Pelletier (2006) présente un modèle à corrélations dynamiques basé sur une extension du modèle de mélange de distribution : l'approche Markov-switching (Regime Switching for Dynamic Correlation, RSDC). L'approche Markov-switching ajoute à la variable indicatrice une dimension temporelle, en supposant que cette dernière suive une chaîne de Markov homogène de degré un. La structure latente permet alors d'attribuer l'appartenance à un régime conditionnellement à l'information passée. A l'instar du modèle STCC, les corrélations du modèle RSDC évoluent entre des matrices de corrélations constantes dans le temps de taille $K \times K$ ou $R_n \neq R_{n'}$ pour $n \neq n'$. Cette spécification assez lourde en paramètres est complétée dans l'article de Pelletier (2006) par une version parcimonieuse. La contribution du chapitre 2 de cette thèse présente un cas particulier du RSDC basé sur une structure hiérarchique Markovienne cachée (HRSDC).
L'approche Markov-switching est également appliquée par Billio and Caporin (2005) pour introduire un changement de régime dans l'équation (1.19) :

$$Q_t = (1 - a_{s_t} - b_{s_t})\overline{Q}_{s_t} + a_{s_t}\varepsilon_{t-1}\varepsilon'_{t-1} + b_{s_t}Q_{t-1} \tag{1.35}$$

où a_{s_t}, b_{s_t} et $\overline{Q}_{s_t}$ dépendent des valeurs de la variable aléatoire s_t. Dans la chapitre (3), nous présentons également une extension de modèle DCC basée sur une structure factorielle cachée.

Grandes matrices de corrélations conditionnelles Malgré les efforts pour limiter le nombre de paramètres les spécifications des modèles DCC restent au mieux délicates à estimer voire inapplicables pour la modélisation de grandes matrices de corrélations. Ainsi, un axe de recherche se focalise sur des spécifications permettant de prendre en compte plusieurs centaines de séries financières. Palandri (2009) propose une solution dont l'idée est de décomposer un problème numérique complexe en une somme de problèmes faciles à résoudre. Son approche, intitulée Sequential Conditional Correlation (SCC) est construite sur une expression séquentielle de la matrice de corrélation, en séparant les corrélations et les corrélations partielles :

$$R_t = [\prod_{i=1}^{K-1}\prod_{j=i+1}^{K} K_{i,j,t}][\prod_{i=1}^{K-1}\prod_{j=i+1}^{K} K_{i,j,t}]' \tag{1.36}$$

où matrice $K_{i,j,t}$ est une matrice triangulaire inférieure dont les éléments sont :

$$K_{i,j,t}[row, col] = \begin{cases} \rho_{i,j,t} \text{ si } row = j \text{ et } col = i \\ \left(1 - \rho^2_{i,j,t}\right)^{1/2} \text{ si } row = j \text{ et } col = j \\ I_{[row,col]} \text{ sinon} \end{cases}$$

et I est la matrice identité. Ce modèle n'impose pas de forme particulière pour la formulation des corrélations et n'importe quelle spécification peut être utilisée [9].
L'idée de décomposer un problème compliqué en une somme de problèmes plus simples est également exploitée par Engle (2008) avec la méthode *MacGyver*. L'idée est d'estimer chaque corrélation séparément et ensuite de retenir la médiane de ces estimateurs.
Une autre stratégie est déployée par Engle and Kelly (2009) avec le modèle Dynamic Equicorrelation (DECO). Cette formulation suppose que toutes les paires de rendements possèdent la même corrélation à l'instant t, mais que cette corrélation change à chaque période. Dans ce modèle, la matrice de corrélation s'écrit :

$$R_t = (1 - \rho_t)\mathrm{I}_K + \rho_t \mathrm{J}_{K \times K} \tag{1.37}$$

9. Palandri (2009) expose d'ailleurs dans son analyse une spécification originale pour le processus des corrélations basée sur la transformation de Fisher.

avec $\mathrm{I_K}$ une matrice identité de taille K et $\mathrm{J_{K\times K}}$ une matrice carrée de un de taille K. Les équicorrélations sont représentées par la variable ρ et se calculent comme :

$$\rho_t = \frac{1}{K(K-1)} \sum_{i \neq j} \frac{q_{ij,t}}{\sqrt{q_{ii,t} q_{jj,t}}} \tag{1.38}$$

où $q_{ij,t}$ est un élément de la matrice de covariance Q_t issue d'un modèle DCC. Outre son hypothèse simplificatrice d'équicorrélation, l'expression des corrélations définit par l'équation (1.37) présente un avantage numérique dans la mesure où l'inverse et le déterminant de R_t s'écrirent sous une forme analytique [10].

Structures par arbre de décision Plusieurs approches basées sur un arbre de décision ont été suggérées. Barone-Adesi and Audrino (2006) élaborent une structure par arbre développée précédemment dans le cadre univarié (modèle Tree-GARCH, voir Audrino and Buhlmann (2001) et Audrino (2006)).

$$R_t = \left(1 - \sum_{k=1}^{N} \lambda_k \mathbb{1}_{\{(\overline{\rho}_{t-1}, r_{t-1}) \in \mathscr{R}_i\}}\right) \overline{Q}_{t-p}^{t-1} + \left(\sum_{k=1}^{N} \lambda_k \mathbb{1}_{\{(\overline{\rho}_{t-1}, r_{t-1}) \in \mathscr{R}_i\}}\right) \mathrm{I}_d \tag{1.39}$$

où $\lambda_k \in]0,1] \forall k$, $\overline{Q}_{t-p}^{t-1}$ est une matrice de covariance non conditionnelle sur un horizon de p périodes. La partition est effectuée dans la plan $(\overline{\rho}_{t-1}, r_{t-1})$ où $\overline{\rho}_{t-1}$ est une moyenne pondérée des éléments de la matrice de corrélation variant dans le temps. Ce modèle nommé Rolling Window Tree Averaging Conditional Correlations (RW-TACC) nécessite un faible nombre de partitions pour être exploitable en pratique. Cette approche est par la suite perfectionnée dans un cadre non paramétrique par Audrino (2006). Par ailleurs, une autre spécification, toujours inspirée de la méthode développée pour le Tree-GARCH, est appliquée par Trojani and Audrino (2005). Dans ce modèle un Tree-GARCH est appliqué pour extraire les volatilités individuelles et une structure par arbre similaire est appliquée pour le processus des corrélations. Dans ce cas, la structure par arbre ne concerne pas seulement les corrélations mais aussi les dynamiques des volatilités univariées.

Dellaportas and Vrontos (2007) développent également une structure par arbre de décision binaire établissant un lien entre la volatilité individuelle et les corrélations.

Le chapitre 4 de cette thèse propose également un modèle par arbre. A la différence des spécifications précédentes, basées sur une approche déterministe des arbres de décision, nous proposons une approche stochastique basée sur une extension du modèle HMM.

Ainsi, suite aux articles pionniers de Tse and Tsui (2002) et de Engle and Sheppard (2001) introduisants une dynamique des corrélations dans les modèles à corrélations conditionnelles, de nombreux développements ont été proposés. Nous venons d'évoquer dans cette section la plupart d'entre elles. Il n'en reste pas moins que ce bref survol de la littérature reste loin d'être exhaustif. De nombreux compléments figurent dans le survey de Bauwens, Laurent, and Rombouts (2006), qui présente une typologie des GARCH multivariés, ainsi que l'article de synthèse de Silvennoinen and Teräsvirta (2009). La figure 1.1 présente une cartographie des différentes extensions du modèle à corrélations conditionnelles dynamiques en prenant comme point de départ les approches *historiques* de Tse and Tsui (2002) et de Engle and Sheppard (2001).

10. Dans le cas de grands systèmes de corrélation, l'inversion de R_t à chaque instant t quand cette dernière n'a pas d'expression analytique simple pose des problèmes numériques qui se traduisent par des durées d'estimation longues. La possibilité d'avoir l'expression analytique de R_t^{-1} est donc un avantage non négligeable.

Changement de régime :
- *VC MTGARCH* : Multivariate Threshold Varying Conditional Correlations Model, Kwan, Li, and Ng (2010)
- *AT MTGARCH* : Aggregate Threshold DCC, So and Yip (2010)
- *STCC* : Smooth Transition Conditional Correlation, Silvennoinen and Teräsvirta (2005)
- *DSTCC* : Double Smooth Transition Conditional Correlation, Silvennoinen and Teräsvirta (2009)
- *RSDC* : Regime Switching for Dynamic Correlation, Pelletier (2006)
- *HRSDC* : Hierarchical RSDC, chapitre 2 de cette thèse
- *MS-DCC* : Markov-switching DCC, Billio and Caporin (2005)
- *FHM-DCC* : DCC with Factorial Hidden Markov decompostion, chapitre 3 de cette thèse
- *Mix-DCC 1* : Mixture DCC (approche Engle), Putintseva (2010)
- *Mix-DCC 2* : Mixture DCC (approche Tse-Tsui), Galeano and Ausín (2009)

Ecritures matricielles :
- *F-DCC* : Flexible DCC, Billio, Caporin, and Gobbo (2006)
- *G-DCC* : Generalized DCC, Hafner and Franses (2009)
- *BD-DCC* : Block Diagonal DCC, Billio and Caporin (2009)

Strutures par arbre de décision :
- *RWTA-DCC* : Rolling Window Tree Averaging Conditional Correlations, Barone-Adesi and Audrino (2006)
- *Tree DCC 1* : Tree structured DCC, Trojani and Audrino (2005)
- *Tree DCC 2* : Tree structured DCC, Dellaportas and Vrontos (2007)
- *HMDT DCC* : Hidden Markov Decision Tree based DCC, chapitre 4 de cette thèse

Semi/non-paramétrique :
- *NP DCC* : Non Parametric DCC, Long and Ullah (2005)
- *L DCC* : Local DCC, Feng (2007)
- *SPCC* : Semi Parametric DCC, Franses, Hafner, and van Dijk (2005)

Simplifications :
- *Clus. DCC 1* : Clustered DCC, Zhou and Chan (2008)
- *Clus DCC 2* : Clustered DCC, So and Yip (2009)
- *PS DCC* : Proximity Structured DCC, Caporin and Paruolo (2009)

Asymétrie :
- *ADCC* : Asymmetric DCC, Cappiello, Engle, and Sheppard (2006)
- *AB DCC* : Asymmetric Block DCC, Vargas (2006)
- *AML DCC* : Asymmetric Multivariate Laplace Innovations, Cajigas and Urga (2006)

Grandes matrices de corrélations :
- *SCC* : Sequential Conditional Correlation, Palandri (2009)
- *DECO* : Dynamic Equicorrelation, Engle and Kelly (2009)
- *MacGyver DCC* : MacGyver estimation based DCC, Engle (2008)

FIGURE 1.1 – Cartographie des différents axes de développements des modèles à corrélations conditionnelles intégrant une spécification dynamique.

1.3 Cadre général des modèles graphiques probabilistes.

1.3.1 Origines et formalisme

Les modèles graphiques probabilistes (MGP), aussi appelés réseaux bayésiens, représentent un formalisme issu de la recherche en intelligence artificielle. Quoique trouvant ses origines dans les années soixante, cette approche a connu un développement important dans les années quatre-vingt dix afin de modéliser des problèmes complexes provenant de la reconnaissance d'écriture manuscrite, de la séparation de sources auditives, de la reconnaissance d'images ou encore de la classification de données.
Ce formalisme repose sur deux disciplines : la théorie des probabilités et la théorie des graphes [11]. Dans ce cadre, l'impact d'une variable, d'un système, d'un évènement sur un/une autre est représenté par un graphe. Cette représentation graphique de la causalité permet de modéliser une relation de dépendance entre ces variables ; cette relation est alors quantifiée avec des probabilités. Ainsi, plus formellement, un MGP peut être représenté par le couple $\{\mathcal{G}, \{p\}\}$ tel que :

- $\mathcal{G} = (X, E)$ est un graphe acyclique orienté dans lequel X représente les noeuds mais aussi des variables aléatoires et E les arcs, permettant de décrire les relations de dépendances entre les variables.
- $\{p\}$ représente un ensemble de lois de probabilités conditionnelles, où chaque élément correspond à la distribution de X_i, $i = 1, ..., n$, conditionnellement à l'ensemble de ses variables parentes.

Un MGP est donc un formalisme possédant une dimension qualitative, avec un graphe acyclique orienté représentant les relations causales du modèle, et une dimension quantitative, représentée par une probabilité conditionnelle.
Les MGP que nous considérons dans cette thèse sont construits sur une hypothèse Markovienne. En d'autres termes, la caractéristique de ce cas particulier de MGP suppose que les variables du modèle à l'instant t dépendent de l'instant $t-1$. Le modèle à chaîne de Markov caché (appelé selon son abréviation anglaise HMM, pour Hidden Markov Model) constitue ainsi le modèle de référence de cette classe particulière de MGP. La section suivante rappelle les principales propriétés de ce modèle.

1.3.2 Modèle de Markov caché

Origines Le modèle HMM trouve ses origines dans la construction des modèles de mélange de lois [12]. L'objectif du modèle de mélanges consiste à estimer une densité de probabilité en supposant que cette densité est un mélange fini de densités. En notant par $\{\mathcal{Y}_t\}_{t\in\mathbb{N}^*} = \{y_1, ..., y_T\} \subset \mathbb{R}^k$ un vecteur d'observations indépendantes de dimension k ayant pour fonction de densité $f(y_t)$ sur $\mathbb{R}^k$, la densité de $\{\mathcal{Y}_t\}$ a pour expression :

$$f(y_t) = \sum_{i=1}^{N} \pi_i f(y_t; \phi_i) \tag{1.40}$$

où $f(y_t; \phi_i)$ désigne une de densité de probabilité de paramètres ϕ_i. Le paramètre $\pi_i \in (0,1)$ correspond à la proportion de la composante i dans le mélange. Pour que la fonction $f(y_t)$ soit une densité de probabilité, la suite positive $(\pi_i)_{i\in\mathbb{N}^*}$ doit alors vérifier :

$$\sum_{i=1}^{N} \pi_i = 1 \tag{1.41}$$

11. "Graphical models are a marriage between probability theory and graph theory." (Jordan (1999)).
12. Voir par exemple l'ouvrage de McLachlan and Peel (2000) pour un exposé général et l'article de Böhning, Seidel, Alfo, Garel, Patilea, and Walther (2007) pour un survey des derniers développements.

L'estimation du modèle de mélange tel qu'il vient d'être présenté est particulièrement délicate pour la simple raison que l'on ne peut affecter les observations aux composantes du mélange. Une solution usuelle à ce problème consiste alors à reconsidérer le modèle de mélange dans un cadre de données incomplètes. Cette méthode associe à chaque vecteur y_t la variable indicatrice $\{\mathscr{S}_t\}_{t\in\mathbb{N}^*} = \{s_1, ..., s_T\} \subset \mathbb{R}$ telle que :

$$\mathbb{1}_{\{s_n=i\}} = 1 \tag{1.42}$$

Le modèle de mélanges en données incomplètes est donc représenté par le couple $\{\mathscr{Y}_t, \mathscr{S}_t\}_{t\in\mathbb{N}^*}$. Les données étant supposées réparties dans différentes composantes, le poids d'une composante peut s'interpréter comme la probabilité qu'une observation appartienne à la composante i : $\pi_i = \mathbb{P}[s_t = i]$. Le modèle est alors construit à partir d'une structure sous-jacente latente formée de la variable indicatrice $iid\ \mathbb{1}_{\{s_n=i\}}$. La probabilité conditionnelle et la probabilité jointe des observations s'écrivent respectivement :

$$\mathbb{P}[y_t|s_t] = \sum_{i=1}^{N} f(y_t;\phi_i)\mathbb{1}_{\{s_n=i\}} \text{ et } \mathbb{P}[y_t, s_t] = \sum_{i=1}^{N} \pi_i f(y_t;\phi_i)\mathbb{1}_{\{s_n=i\}} \tag{1.43}$$

Le modèle de mélanges de lois peut être vu comme un modèle à variable latente en supposant qu'aux données observées correspond une indicatrice d'appartenance à une composante. Cette approche peut être enrichie en supposant que l'indicatrice suit une chaîne de Markov cachée, ce qui permet d'introduire une dimension temporelle sur l'indicatrice des composantes.

Chaîne de Markov Rappelons brièvement les principales définitions et propriétés nécessaires concernant les chaînes de Markov [13]. Soit $\mathscr{S}$ un espace dénombrable appelé *espace d'états*, dans lequel chaque $i \in \mathscr{S}$ est appelé *état*. Une chaîne de Markov d'ordre k est alors définie par une suite de variables aléatoires $(s_t)_{t\in\mathbb{N}}$ à valeur dans $\mathscr{S}$ telle que, pour tout $i_t \in \mathscr{S}$, $t \geq 1$:

$$\mathbb{P}[s_{t+1} = i_{t+1}|s_t = i_t, ..., s_0 = i_0] = \mathbb{P}[s_{t+1} = i_{t+1}|s_t = i_t, ..., s_{t-k+1} = i_{t-k+1}] \tag{1.44}$$

Les modèles de mélange basés sur une chaîne de Markov utilisent généralement une chaîne de Markov d'ordre un [14], qui admet une interprétation intuitive simple : *l'état futur ne dépend que de l'état présent, et pas de l'état passé*. La dynamique est donc entièrement basée sur la probabilité de transition entre les différents états, représentée par :

$$a_{ij} = \mathbb{P}[s_{t+1} = j|s_t = i] \tag{1.45}$$

La figure 1.2 est un exemple de représentation graphique d'une chaîne de Markov à trois états.
Une chaîne de Markov $(s_t)_{t\in\mathbb{N}}$ est alors dite homogène si les probabilités de transition d'un état à un autre sont indépendantes de t :

$$\mathbb{P}[s_{t+1} = j|s_t = i] = \mathbb{P}[s_1 = j|s_t = i] \tag{1.46}$$

Les probabilités de passage d'un état à un autre d'une chaîne de Markov homogène à n états permettent d'établir une matrice de transition :

$$\mathscr{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{1.47}$$

13. Pour un exposé plus complet, nous renvoyons à (Billingsley, 1986, chap. 1.8) and (Shiryaev, 1996, chap. 1.12 and 8).

14. $\Pr[S_{t+1} = s_{t+1}|S_t = s_t, ..., S_0 = s_0] = \Pr[S_{t+1} = s_{t+1}|S_t = s_t]$.

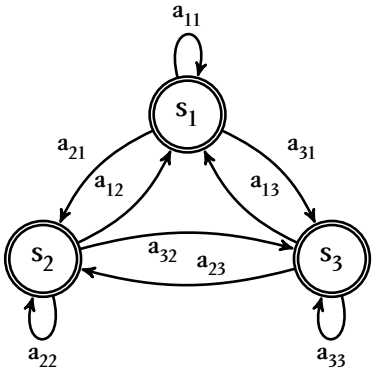


FIGURE 1.2 – Représentation des probabilités de transition d'une chaîne de Markov à trois états.

avec :

$$\sum_j a_{ij} = 1 \text{ et } 0 \leq a_{ij} \leq 1 \tag{1.48}$$

Toute matrice de transition peut être représentée par un graphe orienté valué dont les sommets représentent les états de la chaîne. Ainsi, en supposant que la distribution initiale des états s'écrivent π_t, la distribution à la période $t+1$ s'obtient par : $\pi_{t+1} = \pi_t \mathscr{A}$. Le comportement de la matrice de transition pour des instants supérieurs à 1 s'étudie via le théorème de Chapman-Kolmogorov. La matrice de transition de l'instant 0 à l'instant m s'écrit :

$$\mathscr{A}^{(m)} = \begin{bmatrix} a_{11}^{(m)} & \cdots & a_{1n}^{(m)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{(m)} & \cdots & a_{nn}^{(m)} \end{bmatrix} = \mathscr{A}^m \tag{1.49}$$

où $a_{ji}^{(m)}$ représente la probabilité d'aller de l'état i à l'instant 0 vers l'état j à l'instant m et pour tout $n, k \in \mathbb{N}$ et $h = 1, ..., k-1$:

$$a_{ji}^{(k)} = \sum_{l \in S} a_{li}^{(h)} a_{lj}^{(k-n)} \tag{1.50}$$

Il est alors possible d'identifier différents types d'états en remarquant que l'espace $\mathscr{S}$ peut être partitionné en différentes classes d'états. Pour i et j deux états de la chaîne, on dit que :
– *i mène à j*, noté $i \to j$, si $\mathbb{P}[s_{t+k} = j | s_t = i] > 0$.
– *i* et *j communiquent*, $i \leftrightarrow j$, si $i \to j$ et $j \to i$.
– un état est *récurrent* si $\mathbb{P}[\cup_{n=1}^{\infty}(s_n = i)|s_0 = i] = 1$.
– un état est *transitoire* si $\mathbb{P}[\cup_{n=1}^{\infty}(s_n = i)|s_0 = i] < 1$ [15].
– une chaîne de Markov dans laquelle tous les états communiquent est dite *irréductible*.

Modèle HMM L'idée du modèle à chaîne de Markov caché (HMM) est que l'état du système n'est pas spécifié par l'observation mais grâce à un processus caché qui établit une probabilité de correspondance entre une observation et un état. Chaque observation y_t est donc reliée à une variable aléatoire s_t suivant une chaîne de Markov par une probabilité. L'hypothèse selon laquelle les variables aléatoires associées aux observations suivent une chaîne de Markov d'ordre un implique une dimension temporelle puisque la probabilité d'être dans un état en $t+1$ dépend de l'état en t.

15. La quantité $\mathbb{P}[\cup_{n=1}^{\infty}(S_n = i)|S_0 = i]$ s'interprète comme la probabilité conditionnelle de retour à l'état i sachant que la chaîne est démarrée à l'état i.

La différence principale avec le modèle de mélange indépendant porte donc sur l'indicatrice, qui admet une dépendance temporelle. La variable aléatoire s_t suit généralement (mais pas forcément) une chaîne de Markov d'ordre un mais surtout respecte les conditions d'homogénéité, d'irréductibilité et d'ergodicité. En notant par N le nombre d'états, un modèle HMM correspond au couple $\{\mathscr{Y}_t, \mathscr{S}_t\}_{t\in\mathbb{N}}$. Il est donc caractérisé par trois éléments :

1. un vecteur des transitions initiales $\nu = \mathbb{P}[s_1 = i]\ \forall i \in S$ de taille $(N \times 1)$.
2. une matrice de transition $A = (a_{ij})$ de taille $(N \times N)$ telle que

$$a_{ij} = \mathbb{P}[s_{t+1} = j | s_t = i] \forall i, j \in S \tag{1.51}$$

3. des probabilités d'observations indépendantes conditionnellement aux états cachés :

$$\mathbb{P}[y_t | s_t = i] = f(y_t; \phi_i) \tag{1.52}$$

où ϕ_i correspond aux paramètres d'une famille de loi discrète paramétrique. Dans le cas discret, la loi conjointe suivie par le modèle de Markov caché défini par le couple $\{\mathscr{Y}_t, \mathscr{S}_t\}_{t\in\mathbb{N}}$ a pour expression :

$$\mathbb{P}[y_{1:T}, s_{1:T}] = \mathbb{P}[y_{1:T} | s_{1:T}] \times \mathbb{P}[s_{1:T}] = \prod_{t=1}^{T} \mathbb{P}[y_t | s_t] \times \prod_{t=1}^{T-1} \mathbb{P}[s_{t+1} | s_t] \times \mathbb{P}[s_1] \tag{1.53}$$

Pour représenter le modèle HMM graphiquement sous forme de réseaux bayésiens dynamiques (Dynamic Bayesian Network, DBN), la convention graphique consiste à représenter le modèle par tranche de temps t successives, comme c'est le cas sur la figure 1.3.

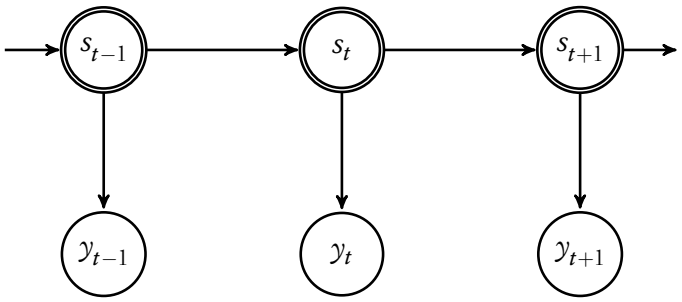


FIGURE 1.3 – Représentation d'un HMM sous forme de DBN.

1.3.3 Extensions

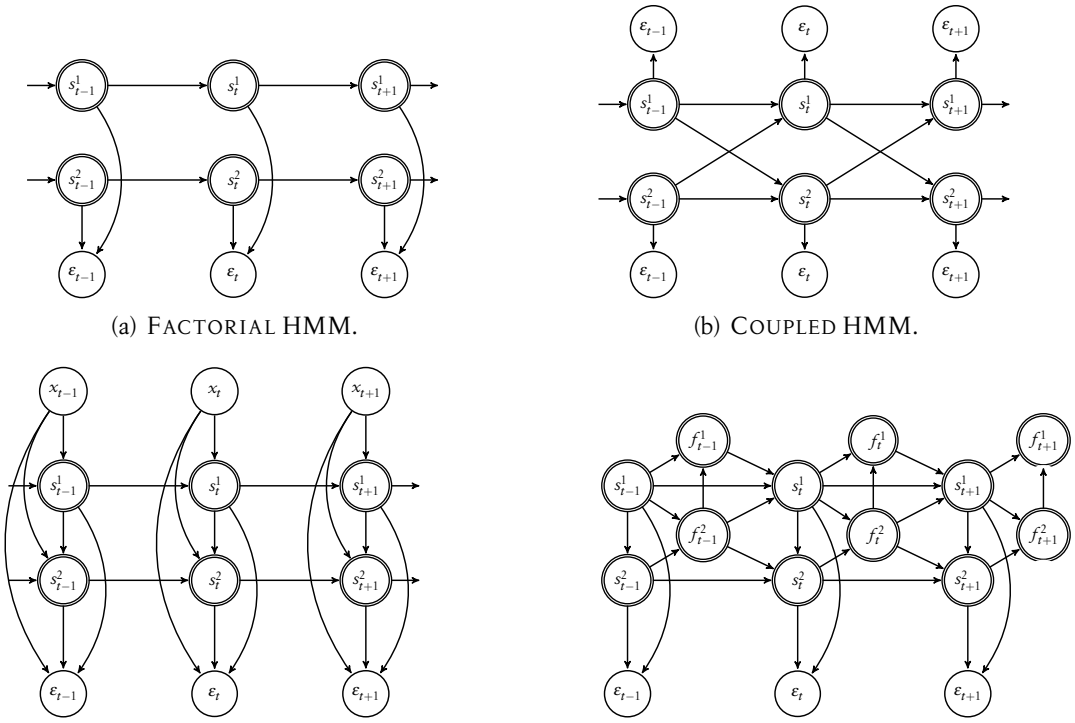
Le modèle HMM constitue l'un des plus simple MGP. De nombreuses extensions sont apparues dans la littérature, répondant à des besoins spécifiques de modélisation. Nous présentons ici celles que nous avons utilisé dans les contributions de cette thèse.

HMM factoriel Introduit par Ghahramani and Jordan (1997), le HMM factoriel (FHMM) suppose que l'état caché est décomposé en plusieurs états sous une forme factorisée. La chaîne de Markov principale du modèle est donc décomposée en plusieurs chaînes évoluant en parallèle. Une représentation graphique du FHMM sous forme de réseau bayésien dynamique (DBN) est visible sur la figure 1.4(a), où $\{s^1\}$ et $\{s^2\}$ sont deux chaînes de Markov évoluant en parallèle.

HMM couplé Le HMM couplé (Coupled HMM, CHMM) est exposé en détail dans l'étude de Brand (1997). Le modèle est bâti sur l'hypothèse qu'une chaîne de Markov peut avoir une influence sur une autre chaîne de Markov du modèle. Cette extension relâche donc l'hypothèse d'indépendance des chaînes du modèle FHMM, comme l'illustre la figure 1.4(b). Dans cette $\{s^1\}$ et $\{s^2\}$ sont deux chaînes de Markov évoluant en parallèle mais possédant des interdépendances.

Arbre de décision Markovian Jordan, Ghahramani, and Saul (1997) proposent un arbre de décision stochcastique (Hidden Markov Decision Tree, HMDT). Ce modèle est un fait une spécification hybride puisque qu'un HMDT est un HMM qui est factorisé et couplé. C'est donc le mariage entre le FHMM et le CHMM. Ce modèle est représenté sur la figure 1.4(c). Ce modèle autorise la présence d'un *input*, représenté par la variable x_t, pour conditionner les dépendances de l'arbre.

HMM hiérarchique Le HMM hiérarchique (Hierarchical HMM, HHMM) est une extension du modèle HMM de base introduite par Fine, Singer, and Tishby (1998). L'idée est de construire un processus stochastique à plusieurs niveaux en adoptant une structure par arbre afin d'obtenir un entrelacement de régimes. Une représentation sous forme de DBN est donnée sur la figure 1.4(d). La composante f_t^i, $i = 1, 2$, du graphique est une variable indicatrice prenant la valeur un ou zéro suivant la profondeur de l'arbre nécessaire pour produire une observation. Il est peut être nécéssaire de préciser que la représentation DBN n'est certainement pas la représentation la plus intuitive qui soit pour représenter ce modèle. Une représentation sous forme d'arbre est visible dans le chapitre 2.



(a) FACTORIAL HMM. (b) COUPLED HMM.

(c) HIDDEN MARKOV DECISION TREE. (d) HIERARCHICAL HMM.

FIGURE 1.4 – Quatre extensions de HMM utilisées dans cette thèse, où la variable s_t représente une chaîne de Markov, ε_t les observations, x_t une variable d'input et f_t une variable indicatrice.

Cette présentation des MGP et des extensions HMM présentée dans cette section est volontairement brève. Nous nous sommes limités à présenter les concepts généraux nécessaires à la lecture de cette thèse. Le lecteur intéressé trouvera de nombreux développements, notamment concernant l'apprentissage de ces modèles et l'inférence dans les ouvrages de Jordan (1999) et Bishop (2006). L'article de Ghahramani (1998) offre un exposé concis de ces modèles tandis que la thèse de Murphy (2002) explicite d'une manière exhaustive la théorie des MGP. Une présentation des modèles HMM est présentée dans l'incontournable article de Rabiner (1989) ; la monographie de Cappé, Moulines, and Ryden (2005) fournit un exposé théorique complet du modèle HMM.

1.4 Contributions de cette thèse

Les contributions de cette thèse se placent dans la littérature des modèles GARCH multivariés à corrélations conditionnelles dynamiques. L'apport des travaux qui y sont présentés concerne la façon de modéliser les changements de régimes avec des modèles à chaîne de Markov cachée. En effet, l'approche Markov-Switching [16] introduite par Lindgren (1978) en économie puis fortement popularisée par Hamilton (1989), a donné lieu à une abondante littérature [17].
Une caractéristique commune aux travaux de cette littérature est l'utilisation du modèle de base, à savoir le HMM. Dans cette thèse, nous avons considéré le modèle de base comme trop restrictif. Aussi, les trois modèles que nous présentons sont construits à partir d'extensions du HMM issues des travaux du domaine des MGP.

Contribution du chapitre 2 Ce chapitre présente un cas particulier du modèle RSDC Pelletier (2006) appelé Hierarchical-RSDC (HRSDC). Il est construit avec une structure markovienne cachée hiérarchique (HHMM) introduite par Fine, Singer, and Tishby (1998). Le gain de cette approche par rapport au Markov-Switching classique est d'augmenter la granularité des régimes. Le modèle HRSDC est comparé au modèle DSTCC, de classe STAR, présenté par Silvennoinen and Teräsvirta (2009) dont il est l'analogue dans la classe Markov-Switching. Nous présentons également deux applications : une première sur données simulées et une seconde sur des données réelles.

Contribution du chapitre 3 Cette seconde contribution présente un modèle à changement de régime basée de la spécification de Engle and Sheppard (2001). Cette approche avait déjà été formulée

16. Markov-switching et HMM sont deux termes renvoyant au même concept. Cependant, ces deux dénominations ne sont pas encore vraiment standardisées et un modèle Markov-switching sera souvent qualifié de HMM (voir Cappé (2005, chap. 1.2).

17. Citons, sans être exhaustif, le travail de Hamilton (1990) pour la mise au point d'un filtre d'estimation des probabilités de transition, ainsi que Krolzig (1997) pour une extension très complète du modèle original au cadre multivarié. Kim (1994) introduit un filtre de lissage des probabilités affinant le calcul des probabilités a posteriori.

Le premier modèle ARCH avec Markov-switching (SWARCH) est proposé par Hamilton and Susmel (1994) ; une première généralisation au GARCH (GRS) apparaît dans une analyse des taux d'intérêts courts de Gray (1996). L'innovation du modèle développé par Gray réside dans l'écriture de la variance conditionnelle. Pour le modèle SWARCH, chaque variance conditionnelle dépend non seulement du régime, mais aussi de tout l'historique [18]. Dans le modèle GRS, la variance conditionnelle dépend seulement du régime courant en intégrant l'espérance de la variance passée. Klaassen (2002) améliore ce dernier modèle en élargissant le spectre de l'information pour la variance retardée. Haas, Mittnik, and Paollela (2004b) proposent une approche un peu différente avec le modèle MSG(N) qui consiste à estimer N modèles GARCH parallèlement et à sauter d'une variance à une autre.Ils contournent ainsi le problème de path-dependancy. Cette solution généralise l'approche GARCH avec Markov-switching et plusieurs modèles, comme ceux de Haas, Mittnik, and Paollela (2004a) et de Franq, Roussignil, and Zakoian (2001) apparaissent comme des cas particuliers. Enfin, Liu (2006) finalise le modèle MSG(N) en étudiant ses conditions de stationnarités avec des hypothèses moins restrictives que celles de Haas, Mittnik, and Paollela (2004b).

Les modèles GARCH multivariés basés sur une approche Markov-switching ont été évoqué dans la section 1.2.2.

par Billio and Caporin (2005). Dans ce modèle, les corrélations évoluent entre deux régimes, dont le passage de l'un à l'autre s'effectue via une chaîne de Markov cachée. Dans cette configuration, tous les éléments de la matrice des corrélations conditionnelles sont alors soumis à la même dynamique de saut. Dans ce chapitre, cette hypothèse est considérée comme trop restrictive et nous permettons à tous les éléments de la matrice de corrélation d'avoir leur propre dynamique de saut. Cette possibilité de dynamique des corrélations par paire est réalisée en utilisant une structure Markovienne factorisée, proposée par Ghahramani and Jordan (1997).

Contribution du chapitre 4 Cette dernière contribution présente un arbre de décision mettant un établissant un lien entre les volatilités individuelles et les corrélations. Diverses modélisations par arbre de décision ont déjà été proposées dans le domaine des modèles à corrélations conditionnelles. Cependant, ces modèles adoptent tous une structure déterministe. Dans ce travail, nous utilisons une structure stochastique provenant de l'approche développée par Jordan, Ghahramani, and Saul (1997), le modèle HMDT. Dans un premier temps, nous utilisons une structure Markovienne factorisée afin de discriminer entre des régimes de faibles et de fortes volatilités ainsi qu'entre un régime de faibles ou fortes corrélations. Le lien permettant de relier des états spécifiques de volatilité à un état spécifique de corrélations est alors réalisé en couplant les chaînes du modèle factorisé.

CHAPITRE 2

Hierarchical Hidden Markov Structure for Dynamic Correlations : the Hierarchical RSDC Model.

2.1 Introduction [1]

Since the seminal papers of Engle (2002) and Engle and Sheppard (2001), the study of multivariate GARCH with dynamic correlations has given rise to many extensions and developments. This growing interest in the subject has been engendered by several empirical studies about stock market behaviors (like Longin and Solnik (1995, 1996, 2001)). They show that the hypothesis of constant correlations (CCC model of Bollerslev (1990)) is not realistic. Although, if the assumption of dynamic correlations is now widely accepted in the literature, the matter of the form of the dynamic remains an open question ; see Bauwens, Laurent, and Rombouts (2006) or Silvennoinen and Teräsvirta (2009) for recent surveys.

The purpose of this paper is to present a new multivariate GARCH model with dynamic correlations. We propose a regime switching model that is part of the Markov-Switching class. It is a special case of the RSDC model of Pelletier (2006). This model is halfway between CCC of Bollerslev (1990) and DCC of Engle and Sheppard (2001). Correlations are constant within regime, but vary from one regime to another and the transition between the different regimes are performed by a Markov chain. Silvennoinen and Teräsvirta (2005) have proposed a model with smooth transition between regime for correlations (STCC model), which can be seen, under certain assumptions, as the competitive STAR approach of Pelletier's model. The STCC requires the correlations to varying between two matrices of constant correlations. The transition between this two extreme matrices is governed by a conditional logistic function. Recently, they built an extension of this model, the Double-STCC (see Silvennoinen and Teräsvirta (2007)) in which conditional correlations vary across four matrices of constant correlations through two logistic functions. Our new model, the Hierarchical-RSDC (HRSDC) can be seen as a Markov-Switching version of the DSTCC. In this new model, correlations vary between four correlation matrices constant in time, but their transition from one matrix to another is determined by hierarchical hidden Markov structure. The originality of this structure lie in its ability to establish a hierarchy between the hidden states in order to increase the granularity of the regime. This hierarchical hidden structure was first developed by Fine, Singer, and Tishby (1998) for handwriting recognition. In the present context, this specific structure will allow us to bring out a finer definition of regimes that do not conform to the classical Markov-Switching approach.

The paper is organized as follows. In section 2.2, we briefly give a review of the dynamic conditional correlations models and present a overview of regime switching model, including STAR and Markov-switching approach, applied to correlation modeling. The Hierarchical-RSDC model is introduced in section 2.3.1. Section 2.4 presents results of Monte-Carlo simulations and two empirical applications and compare the HRSDC with the DSTCC and DCC models. Section 2.5 presents concluding remarks and exposes some directions for future research.

2.2 DCC-MGARCH and Regime Switching models

The general framework of multivariate GARCH models with dynamic correlations assumes that a the stochastic process $\mathbf{r}_t$ followed by the observations of size $(K \times T)$ is defined by :

$$\mathbf{r}_t | \mathscr{F}_{t-1} \sim \mathscr{L}(0, H_t) \tag{2.1}$$

where $\mathscr{L}$ is a parametric distribution function with mean equal to zero and conditional variance H_t follows :

$$H_t = D_t R_t D_t \tag{2.2}$$

1. Joint work with Vêlayoudom MARIMOUTOU.

with $D_t = \text{diag}(h_{1,t}^{1/2}, ..., h_{K,t}^{1/2})$, a diagonal matrix composed of the standard deviation of the K univariate series, obtained for example with a GARCH filtering. This filtering permits to calculate the standardized residuals, which are expressed as $\varepsilon_t = D_t^{-1}\mathbf{r}_t$. The expectation of these standardized residuals gives the conditional correlations :

$$\mathbb{E}_{t-1}[\varepsilon_t \varepsilon_t'] = D_t^{-1} H_t D_t^{-1} = R_t \tag{2.3}$$

The first multivariate volatility model taking into account the conditional correlations was the CCC of Bollerslev (1990). It adopts the simplest dynamic ever by taking the conditional correlations constant through time, i.e. $R_t = R \, \forall t$. The correlation matrix is then equal to the unconditional correlation matrix. However, several empirical studies show that correlations among assets are not constant. These results imply that the CCC seems too restrictive and give rise to an extensive literature about the process follows by the conditional correlations.
Tse and Tsui (2002) suggested to express the correlations with an autoregressive form (DCC_T model) :

$$R_t = (1 - \theta_1 - \theta_2)R + \theta_2 R_{t-1} + \theta_2 \Psi_{t-1}, \ \theta_1 + \theta_2 \leq 1 \tag{2.4}$$

where R is the unconditional correlation matrix while the elements of the correlation matrix Ψ_{t-1} are defined as :

$$\psi_{ij,t-1} = \varepsilon_{i,t-1}\varepsilon_{j,t-1} / \sqrt{\varepsilon_{i,t-1}^2 \varepsilon_{j,t-1}^2} \text{ with } 1 \leq i \leq j \leq K \tag{2.5}$$

In the same way, Engle and Sheppard (2001) adopt an autoregressive formulation for R_t (DCC_{ES} model) with :

$$R_t = \text{diag}\{Q_t\}^{-1/2} Q_t \text{diag}\{Q_t\}^{-1/2} \tag{2.6}$$

where Q_t is the conditional covariance matrix such as :

$$Q_t = (1 - \alpha - \beta)\overline{Q} + \alpha \varepsilon_{i,t-k} \varepsilon_{j,t-k}' + \beta Q_{t-l} \tag{2.7}$$

with $\overline{Q}$ the unconditional covariance matrix. These initial formulations define the basic framework of dynamic correlation models and represent the starting point of many extensions ; we refer to Bauwens, Laurent, and Rombouts (2006) and Silvennoinen and Teräsvirta (2009) for recent surveys in this field of research.
In this work, we are interested by two competing approaches to modeling regime switching for the correlations. The first one is the STAR type model of Silvennoinen and Teräsvirta (2005). The second is from Pelletier (2006) and is build in a Markov-Switching setup. Recall that this two classes of models -STAR and Markov-Switching- are built on two ways to treat the problem. The STAR class assumes that the switch between the regimes is deterministic and determinate by a observable transition variable. The Markov-Switching class is quite different. It assumes that the process followed by the observation is governed by an underlying hidden stochastic process. To better understand our proposed model, both approaches are described in more details in the following subsection.

2.2.1 (S)TAR approach

Theoretical aspects The Threshold Autoregressive (TAR) model of Tong and Lim (1980) was one of the first approach to model regime switching. It assumes that at each time t, the regime is determined by a threshold variable and a threshold value. A two regimes model will then be composed by two parallel sub-regimes where the transition from one to another depends of the position of the threshold variable against the threshold value. The SETAR model, for Self-Exciting TAR, is a direct extension in which the threshold variable is the lagged time series itself.

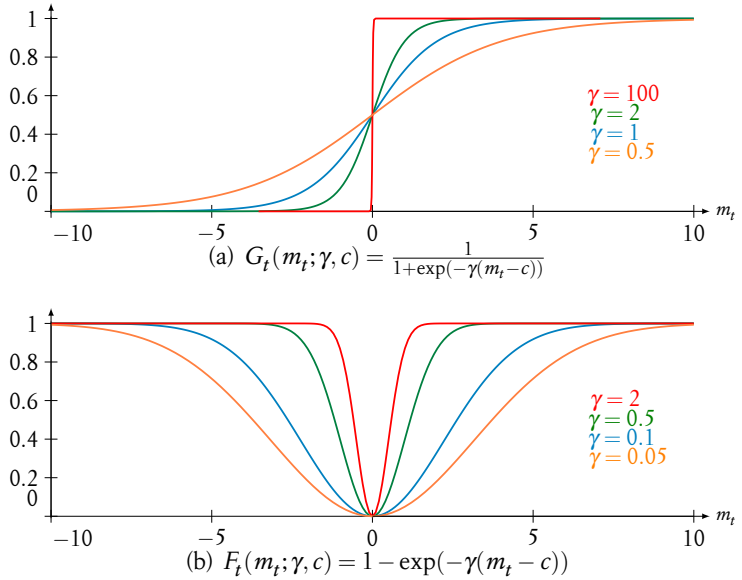


FIGURE 2.1 – Examples of logistic and exponential functions (resp. subfigures 2.1(a) and 2.1(b)) with $c = 0$ and different values of γ.

However, these early models are distinguished by a very sudden switch between regimes. Teräsvirta (1994)) proposes another extension in order to avoid this problem with the Smooth Transition AR (STAR). His solution consists to replace the indicator function by a continuous function $G(y_{t-d};\gamma,c)$ which vary smoothly and monotonically from zero to one. This transition function depends on three variables. The first is the transition variable y_{t-d} ; the two others are parameters which determined the localization (c) and the smoothness (γ). Along all possible functions, the most commonly-used are the logistic function (LSTAR model) :

$$G(y_{t-d};\gamma,c) = (1+\exp(-\gamma\prod_{i=1}^{n}(y_{t-1}-c_i)))^{-1},\ c_1 < \ldots < c_n, \gamma > 0 \tag{2.8}$$

and the exponential function (ESTAR model) :

$$G(y_{t-d};\gamma,c) = 1-\exp(-\gamma\prod_{i=1}^{n}(y_{t-1}-c_i)^2),\ c_1 < \ldots < c_n, \gamma > 0 \tag{2.9}$$

Figure 2.1 shows this two functions for various smoothness values. The main advantage of the STAR model is the effect of the smoothness parameter. It provides a strong transition when $\gamma \to +\infty$ and the STAR can be seems as a TAR. Consequently, when $\gamma \to 0$, the transition is very smooth and when $\gamma = 0$, the STAR performed as a linear model (see van Dijk, Teräsvirta, and Franses (2000) for a complete survey, or Franses and van Dijk (2004)).
These methodologies, first developed for autoregressive processes, were quickly applied to GARCH models. Tong's approach has been taken over by Glosten, Jagannathan, and Runkle (1993) for the GJR-GARCH model, in which the conditional variance follows two different processes depending on the sign of error terms. In the same spirit, the model TGARCH developed by Rabemananjara and Zakoian (1993)) specifies various regimes for the dynamics of the conditional standard deviation depending on the sign of error terms. Finally, Li and Li (1996) develop the Double-Threshold ARCH (DTARCH), which imposes a threshold structure for conditional variance and also for the conditional mean. Following the example of the latter model, Koutmos (1998) proposes a dual-threshold model in order to test the asymmetrical response of mean and conditional variance with

past information (asAR-TGARCH model). The STAR approach also results in various solutions capable of taking into account the asymmetry. Hagerud (1996) makes two STAR-GARCH models. The first type is LSTAR where the transition variable is based on the sign of residuals. The second model is based on a smoothing ESTAR and the transition variable is the absolute value of residuals. These two models are complemented by the analysis of Gonzales-Riviera (1998) with take as transition variable the asymmetry parameter of the variance. Lundbergh and Teräsvirta (1998) build the STAR-STGARCH model in which the mean follows a STAR process and the conditional variance follows a STAR-GARCH. This approach extends two other nonlinear approaches that are GJR-GARCH and Quadratic GARCH of Sentana (1995). Nam, Pyun, and Arize (2002) apply a non-linear structure based on the STAR approach to the mean and conditional variance to capture the asymmetry (ANST-GARCH model). There is also the approach of Lanne and Saikkonen (2005), in which the transition variable is the conditional variance and whose goal is to model series with high persistence in the conditional variance. Finally, Medeiros and Veiga (2004) establish a general formulation with the Flexible Coefficient Smooth Transition GARCH model (FC-STGARCH), encompassing most of the models mentioned above.

STCC model The construction of a STAR structure for the process followed by the correlations of a multivariate GARCH model was produced by Silvennoinen and Teräsvirta (2005) and Berben and Jansen (2005), leading to the Smooth Transition Conditional Correlation GARCH model (STCC-GARCH). The idea is to establish a conditional correlation matrix R_t as a combination of two extremes correlation matrices R_1 and R_2 constant over time. The transition from one to another is provided by the logistic function as defined in the equation (2.8). Specifically, the STCC specifies dynamic of the correlations as :

$$R_t = (1 - G_t)R_1 + G_t R_2 \text{ with } G_t = (1 + e^{-\gamma(m_t - c)})^{-1} \tag{2.10}$$

where γ is the smoothing parameter, c is setting the localization and m_t is the transition variable. To ensure the semi-positivity of the conditional matrix R_t, the matrix R_1 and R_2 must themselves be positive definite, via a Cholesky decomposition for example. The transition variable m_t can be both deterministic or stochastic, and its choice will remain crucial. If there is no real consensus to choice this variable, the study of empirical analysis using the STCC model shows two types of transition variable. It can be :

- a endogenous or exogenous variable. In Silvennoinen and Teräsvirta (2005), the authors apply the model STCC to study the correlations of assets from the daily S&P500. The transition variable is the value of the lagged returns of the squared S&P500. Their strategy is based on the idea that a strong difference value of the index at a given time represents a strong turbulence of the assets of the S&P500 to the next period.
- calendar time, i.e. $m_t = t/T$. This specification results in the terminology of Silvennoinen and Teräsvirta (2005) to the Time-Varying Smooth Transition Conditional Correlation (TVSTCC) model. That's the choice of Berben and Jansen (2005) in order to study correlations of equity returns.

Silvennoinen and Teräsvirta establish a strategy of inference for choosing the best transition variable. The idea is to find the most explicative variable controlling the dynamics of correlations. In this context, the transition variable varies depending on the purpose of the analysis. In the case where preliminary analysis did not reach to discriminate clearly this transition variable, the choice of calendar time seems to be the most consensual.

Silvennoinen and Teräsvirta (2007) extended the STCC model by introducing two transition functions to evolve correlations between four correlation matrices constant over time. The conditional

correlations process then takes the following form :

$$R_t = (1-G_{1t})R_{(1)t} + G_{1t}R_{(2)t} \text{ with } \begin{cases} R_{(i)t} = (1-G_{2t})R_{(i1)} + G_{2t}R_{(i2)} \\ G_{it} = (1+\exp(-\gamma(m_{it}-c_i)))^{-1} \end{cases}, i = 1,2 \tag{2.11}$$

As for the STCC model, the transition variable can be of various kinds (endogenous or exogenous). We now focus on a special case of the DSTCC, in which transition functions are $m_{1t} = m_{2t} = t/T$. In this version, the correlations dynamic can be rewritten in a more illustrative way as :

$$R_t = (1-G_{2t})\underbrace{((1-G_{1t})R_{11} + G_{1t}R_{12})}_{R_{1t}} + G_{2t}\underbrace{((1-G_{1t})R_{21} + G_{1t}R_{22})}_{R_{2t}} \tag{2.12}$$

In this formulation, the correlation matrix is a linear combination of two conditional matrices R_{1t} and R_{2t}. These two matrices are themselves a linear combination of two extreme matrices constant over time and semi-positive definite. Estimation is done by maximizing the likelihood with the usual iterative optimization methods (i.e. Gradient), in one or two steps in accordance with the sample size (see Engle and Sheppard (2001)). However, as often in the regime switching models, the existence of numerous local maxima requires to test many initial conditions.

2.2.2 Markov-Switching approach

Theoretical aspects The Markov-Switching approach has its origins in the construction of mixture models [2]. The objective of these models is to estimate a density assuming that the density is a finite mixture of densities, i.e. a convex combination of densities. In this model, a latent indicator variable is use to attribute an observation to a component. The Markov-switching approach enhances the mixture model by assuming that the indicator follows a hidden Markov chain, which allows us to introduce a temporal dimension on the indicator components. Thus, each observed variable ε_t has a specific relation with a hidden state s_t which follow a first order Markov chain. The switch from one state to another is specified by transition matrix. This linking relation is generally a first order Markov chain, that means the state s_t is independent of state lagged state except s_{t-1} :

$$\mathbb{P}[s_{t+1} = i | s_t = j] = p_{ij} \tag{2.13}$$

These probabilities are grouped together to make the transition matrix of size $N \times N$, which can be expressed as :

$$\mathrm{P} = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} \tag{2.14}$$

with $\sum_{j=1}^{N} a_{ij} = 1$ for each $i = 1, ..., N$. In its more general representation, any Markov-switching model can be written as :

$$\varepsilon_t | \mathscr{F}_{t-1} \sim f(\varepsilon_t; \theta) \tag{2.15}$$

with $\mathscr{F}_{t-1}$ denotes the information set at time $t-1$, θ the parameter set. The density function $f(\varepsilon_t; \theta)$ given past observations over N states of the Markov chain can then be expressed as :

$$f(\varepsilon_t; \theta) = \sum_{n=1}^{N} f(\varepsilon_t | s_t = n; \theta_n)\mathbb{P}[s_t = n; \theta_n] \tag{2.16}$$

2. See, for example, McLachlan and Peel (2000) for a briefing and Böhning, Seidel, Alfo, Garel, Patilea, and Walther (2007) for a survey of recent developments.

with θ_n the parameters of each regime n for $n = 1, ..., N$.
Partly introduced by Lindgren (1978) in economics and highly popularized by Hamilton (1989), the Markov-Switching approach has given rise to a huge literature. We can quote, without being exhaustive, the work of Hamilton (1990) for the development of a filter to estimate the transition probabilities, as well as Krolzig (1997) for a comprehensive extension of the original model to the multivariate framework. Kim (1994) also introduces a smoothing filter refining the calculation of posterior probabilities.
The first ARCH model with Markov-switching (SWARCH) is proposed by Hamilton and Susmel (1994) while the first GARCH extension (GRS) appears in an analysis of the interest rates by Gray (1996). The innovation for the model developed by Gray lies in writing the conditional variance. Indeed, for SWARCH, each conditional variance depends not only of the regime, but on all history [3]. Whereas in the GRS, the conditional variance depends only on the current regime by the expectation of the lagged variance. Klaassen (2002) improves the latter model by broadening the spectrum of information for the lagged variance. Haas, Mittnik, and Paollela (2004b) suggest a slightly different approach with the MSG(N) model, which consists of estimating N GARCH models in parallel and switching directly from a variance to another. They then avoid the problem of path-dependency. It generalizes several Markov-switching GARCH approaches like the models proposed by Haas, Mittnik, and Paollela (2004a) and by Franq, Roussignil, and Zakoian (2001), which appear as special cases. Finally, Liu (2006) deepens the MSG(N) model by studying the conditions of his steady states with less restrictive assumptions than Haas, Mittnik, and Paollela (2004b).

RSDC model The literature on Markov-Switching GARCH models in a multivariate framework is actually much less dense than in the univariate context. Lee and Yoder (2007)) apply the Gray's method to build a bivariate Markov-switching BEKK model. The multivariate GARCH model of Pelletier (2006) is the first, and for now the only one, to adopt a Markov-Switching structure for the correlation process. By imposing constant correlations in each regime and establishing the switch of each other through a Markov chain of order one, this model falls between the CCC and the DCC. Formerly, the RSDC model assumes that the conditional correlation matrix has the form :

$$R_t = \sum_{n=1}^{N} \mathbb{1}_{\{s_n=i\}} R_n \tag{2.17}$$

where $\{s_t\}_{t\in\mathbb{N}}$ is a sequence follows a homogeneous first order Markov chain with N states. R_n is a conditional correlation matrix of size $K \times K$ where $R_n \neq R_{n'}$ for $n \neq n'$. The distribution of the process defined by $\{s_t\}_{t\in\mathbb{N}}$ is determined by a transition matrix written A. So that the conditional correlation matrix is indeed a correlation matrix [4], it is necessary to impose a PSD constraint on the matrix corresponding to the regimes. This can be a constraint Choleski decomposition, i.e. $R_n = C_n C_n'$ with C_n a lower triangular matrix and imposing that diagonal elements of C_n to be written as :

$$c_{j,j} = \sqrt{1 - \sum_{i=1}^{j-1} c_{j,i}^2} \tag{2.18}$$

Pelletier also proposes a more parsimonious specification of the RSDC with the following expression :

$$R_t = \bar{Q}\lambda_{s_t} + \mathbf{I}_K(1 - \lambda_{s_t}) \tag{2.19}$$

3. It is the well-known path-dependency problem.
4. Recall that by definition, for R_t be is a correlation matrix, it must be symmetrical, PSD, have its diagonal elements equal to 1 and the non-diagonal elements must belong to the interval $[-1;1]$.

With two states, the latter expression as being equal to the weighted average of two extreme states. The case $\lambda_{s_t} = 0$ is where the returns are not correlated while $\lambda_{s_t} = 1$ corresponds to the case where they are very correlated. This second formulation requires an identification constraint as follows :

$$\lambda_1 = 1 \text{ with } \lambda_1 > \lambda_2 > ... > \lambda_N \text{ or also } \max_{i \neq j} |R_{ij}| = 1 \text{ with } 1 > \lambda_1 > \lambda_2 > ... > \lambda_N \tag{2.20}$$

Estimation of the RSDC model is done via maximum likelihood, with a nuance according to the specification in question. For the first specification, corresponding to the equation (2.17), Pelletier favors the use of the EM algorithm. The second specification, define by equation (2.19), can be estimated through Gradient methods by exploiting the idea of correlation targeting of Engle and Mezrich (1996) and using Hamilton's filter. In that case, the correlations can be simply calculated as :

$$\mathbb{E}[R_t] = \bar{Q} \sum_{n=1}^{N} \lambda_n \pi_n + I_K \sum_{n=1}^{N} (1 - \lambda_n) \pi_n \tag{2.21}$$

2.3 Hierarchical Hidden Markov Structure for Dynamic Correlations

As we have seen in section 3.2.1, the DSTCC model of Silvennoinen and Teräsvirta (2007) is an expanded version of the STCC and expresses the conditional correlation matrix as a combination of two extreme matrices that are themselves dynamics. As the authors explain, at the beginning of the sample, the correlations defined by the equation (2.12) will vary between the two states $R11$ and $R12$, and then towards the end of the sample between R_{21} and R_{22}. Couples of matrix (R_{11}, R_{12}) and (R_{21}, R_{22}) define, according to the second transition variable, two major states bounded by two extreme dynamic matrices. By introducing another transition around the first one, the DSTCC works on two *primaries* regimes, themselves build under four *secondaries* regimes. This classification into *primary* and *secondary* regimes refers to the concept of sub-regime. The Markov-Switching approach of Pelletier (2006) doesn't consider the case of sub-regime to establish a hierarchy between states that define sub-regimes, and show the nuances induced by the existence of sub-regimes in the dynamic of the correlations. The idea of the model that we present is that, unlike the basic Markov-Switching, it makes possible to establish a hierarchy between states that define sub-regimes. As a first step, we will present the class of hierarchical hidden Markov models, the structure upon which our model has been built. Next we will explain our model.

2.3.1 Structure

The Hierarchical Hidden Markov Model

The Hierarchical Hidden Markov Model (HHMM) was proposed by Fine, Singer, and Tishby (1998) in order to generalize the HMM model. The idea is to build a stochastic process with several levels by adopting a tree structure to obtain an interlacing of regimes. The hierarchy of the tree is built with *internal states*, which are abstract states (i.e. they do not produce observations). The internal states can lead so-called *emitting states*, which produce observations. Internal and emitting states can also lead to a third type of state which are called *exiting states*. The exiting states allow quitting a level of the tree. Each internal state produce a sub-HMM, which can also be itself an HHMM. In this framework, the classical HMM is a special case of the HHMM with only one level. The main advantage of the HHMM compared to the HMM lies in its ability to improve the *granularity* of the regimes. The HHMM approach permits one to break up a time series into several types of regimes. For example, an HHMM with two levels has two types of regimes : the firsts are called *primaries regimes* and the seconds *secondaries regimes*. The combination of the

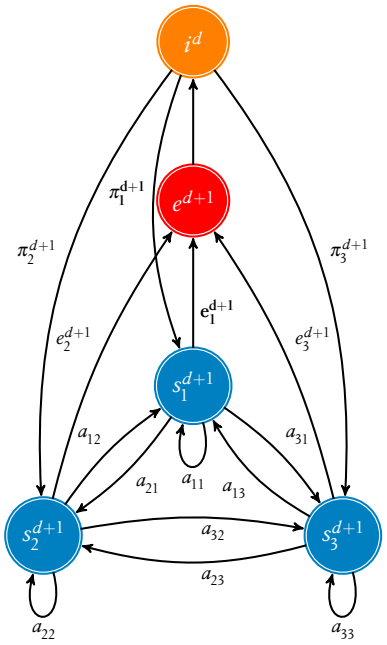


FIGURE 2.2 – Basic structure of an HHMM with three emitting states.

secondaries permits to deduce the primaries ones. This increased granularity allows us to bring out nuances in regimes, something that is not possible the simplified structure of an HMM.

Formally, an HHMM can be represented as the process $\{\mathscr{Y}_t, \mathscr{Q}_t\}_{t\in\mathbb{N}}$ with :

- $\{\mathscr{Y}_t\}_{t\in\mathbb{N}}$ is the process followed by the observations, which are supposed to be conditionally independent to the hidden states.
- $\{\mathscr{Q}_t\}_{t\in\mathbb{N}}$ is a homogeneous first order Markov chain. Each state of an HHMM q_i^d belongs to the set $\mathscr{Q} = \{\mathscr{S}, \mathscr{I}, \mathscr{E}\}$ where $\mathscr{S}$ is the set of emitting states, $\mathscr{I}$ the set of internal states and $\mathscr{E}$ the set of exiting states. The superscript d corresponds to the index Hierarchy (vertical location) in the tree, with $d \in \{1, ..., \mathscr{D}\}$ and subscript i is its horizontal location.

The tree structure is obtained by imposing an internal state at the root (level $d = 1$) [5]. This initial state then have several descendants which can be internal or emitting states. These sub-internal states can themselves have descendants which can be internal or emitting and so on. The transition from level d to level $d+1$ is provided by the probabilities of vertical transitions through an internal state. The return from level $d+1$ to level d is done with the exiting states and corresponds to a probability of exiting. Internal and emitting states of the same sub-HMM of level d communicate with a transition matrix as in a classical HMM model. An internal state leading to another layer of internal/emitting states is called *parent* state. A parent state leads to *child* states. Finally, three probabilities govern the dynamic of the hidden structure :

- $\mathscr{A}_k^d = (a_k^d(i,j))$: is the matrix of horizontal transition of state i to state j where i and j are two states of the same sub-HMM of level d, i.e. $a_k^d(i,j) = \mathbb{P}[q_{t+1}^d = j | q_t^d = i]$, $q_{t+1}^d, q_t^d \in \{\mathscr{S}, \mathscr{I}\}$.
- $e_i^d = \mathbb{P}[q_{t+1}^d | q_t^{d+1}]$, with $q_t^{d+1}, q_{t+1}^d \in \{\mathscr{S}, \mathscr{I}\}$: vertical probability to leads from a child at level $d+1$ state to parent state at level d.
- $\pi_i^d = \mathbb{P}[q_{t+1}^{d+1} | q_t^d]$ with $q_t^{d+1}, q_{t+1}^d \in \{\mathscr{S}, \mathscr{I}\}$: vertical probability to lead from a parent state to its child state.

5. We will see later that this condition is not always necessary. The HRSDC model has not an internal state at the summit.

To ensure the transition matrix to be stochastic [6], it is necessary to impose that, for each sub-model depending of parent k :

$$\sum_{j \in ch(k)} a_k^d(i,j) + e_i^d = 1 \text{ and } \sum_{i \in ch(k)} \pi_i^d = 1 \tag{2.22}$$

where $i, j \in ch(k)$ are two states with parents k. Figure 2.2 is an illustration of the basic structure of an HHMM and figure A.1 is a complete example (see appendix A.1). As we can seen, the number of parameters of the model depends on the structure of the tree. Xie (2005) uses a useful notation to represent parameters space of an HHMM. Let Q the size of the space state of each sub-model, Xie assume that global configuration of an HHMM starting from the root up to the d^{th} level can be written as :

$$q^{(d)} = \overline{(q^1 q^2 ... q^d)} = \sum_{i=1}^{d} q^i Q^{d-i} \tag{2.23}$$

Thus, assuming that there is only one state at the root, all the parameters comprising an HHMM satisfy the condition :

$$\theta = (\cup_{d=2}^{D} \cup_{i=0}^{Q^{d-1}-1} \{\mathscr{A}_k^d, \pi_i^d, e_i^d\}) \cup (\cup_{i=0}^{Q^{D-1}} \{\phi_i\}) \tag{2.24}$$

where ϕ_i corresponds to the parameters of a parametric probability distribution.

The Hierarchical RSDC model

The objective of our model is to vary the correlations between two extreme major regimes, while allowing the existence of secondaries regimes. As shown by Silvennoinen and Teräsvirta (2007), the correlation process is bounded by four states of constant correlation matrices over time. The structure highlights two primaries regimes, depending on abstract states i_1^1 and i_2^2. Each of these abstract states is connected with emitting states. Thus, the regime corresponding to i_1^1 is determinate by the emitting states s_1^2 and s_2^2 ; that of i_2^1 by s_3^2 et s_4^2. Figure 2.3 shows the hierarchical hidden structure of the Hierarchical RSDC model (HRSDC).

The hierarchical structure allows states to increase the granularity of the regimes. It establishes different types of regimes, which in our case are primaries and secondaries. The primaries regimes correspond to the regimes obtained with a classical Markov Switching model. To a higher level of granularity, these primaries regimes are built with sub-regimes, known as secondaries regimes. The structure allows secondaries's to capture nuances of dynamics that are thinner than the primaries's. The idea of granularity is illustrated on the figure 2.4. Our model has two levels and four secondaries regimes, corresponding to two sub-HMM models.

The pair of emitting states defined by (s_1^2, s_2^2) forms a Markov-Switching model and the same is true for (s_3^2, s_4^2). The link between these sub-models is provided by the abstract states i_1^1 and i_2^1. The model is then built on two sub-models with two emitting states each, which transition matrix are respectively [7] :

$$A_1^2 = \begin{bmatrix} a_{11}^2 & a_{12}^2 \\ a_{21}^2 & a_{22}^2 \end{bmatrix} \text{ and } A_2^2 = \begin{bmatrix} a_{33}^2 & a_{34}^2 \\ a_{43}^2 & a_{44}^2 \end{bmatrix}$$

and verified constraints :

$$\begin{cases} a_{11}^2 + a_{21}^2 + e_1^2 = 1 \\ a_{12}^2 + a_{22}^2 + e_2^2 = 1 \end{cases} \text{ and } \begin{cases} a_{33}^2 + a_{43}^2 + e_3^2 = 1 \\ a_{34}^2 + a_{44}^2 + e_4^2 = 1 \end{cases}$$

6. $\sum_j a_{ij} = 1$ and $0 \leq a_{ij} \leq 1$.

7. Recall that the probability that the state q which was in i at time $t-1$ to be in j at time t is written $\mathbb{P}[q_t = j | q_{t-1} = i] = p_{ji}$.

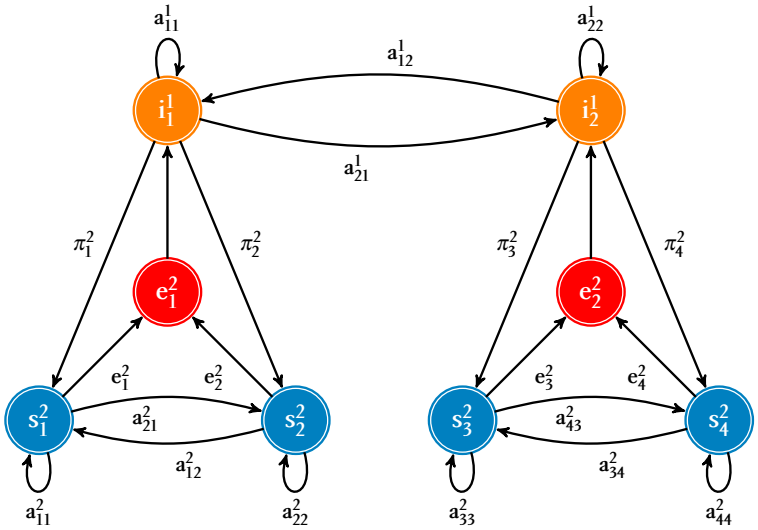


FIGURE 2.3 – Hierarchical Hidden structure of the HRSDC.

where e^2_i, $i = 1,...,4$ is the probability of exiting from a state of level two and go to a parent state at level one. The two sub-HMM communicate via exiting states through abstract states i^1_1 and i^1_2. The activity involved in the transition from one to another of these abstract states is defined by the transition matrix :

$$A^1 = \begin{bmatrix} a^1_{11} & a^1_{21} \\ a^1_{12} & a^1_{22} \end{bmatrix}$$

which verifies :

$$a^1_{11} + a^1_{12} = 1 \text{ and } a^1_{21} + a^1_{22} = 1$$

The parameters π^2_i, $i = 1,...,4$ represent the probability of moving from a parent state of first level to one of its children at the second level. These probabilities must verify :

$$\pi^2_1 + \pi^2_2 = 1 \text{ and } \pi^2_3 + \pi^2_4 = 1$$

The specification for the four correlation matrices constant in time is that outlined by Pelletier (see equations 2.17 and 2.19)[8]. In fact, the only difference between this specification and the RSDC lies in the hierarchical hierarchical hidden structure which led us to view the RSDC as a special case of the HRSDC with only one level. As in the RSDC model, the specification defined by equation 2.17 can be estimate by EM algorithm whereas formulation defined by equation 2.19 allows to use iterative methods like Gradient.

2.3.2 Estimation

Estimation of the HRSDC model is made using the multi-step estimation of Engle and Sheppard (2001) and Engle (2002). This computationally attractive method splits up the log-likelihood $\ell(\varphi, \theta)$ as the sum of two parts : the volatility component $\ell_v(\varphi)$ and the correlations term $\ell_c(\varphi, \theta)$, where φ is the parameters space of the K univariate GARCH model and θ corresponding of the parameters of the correlations. More formally, this is written as :

$$\ell(\varphi, \theta) = \ell_v(\varphi) + \ell_c(\varphi, \theta)$$

8. The four correlation matrices then represent $4K(K+1)/2$ parameters.

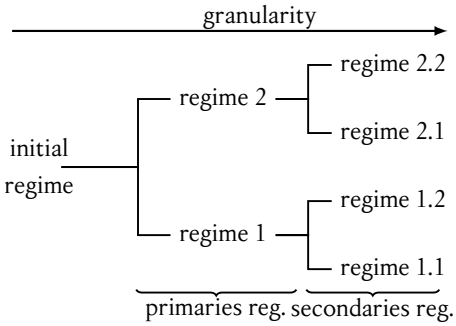


FIGURE 2.4 – The increase of the granularity in the HRSDC model.

with :

- $\ell_v(\varphi|\mathbf{r}_t) = -\frac{1}{2}\sum_{t=1}^{T}(K\log(2\pi) + 2\log(|D_t|) + \mathbf{r}_t' D_t^{-2}\mathbf{r}_t)$
- $\ell_c(\theta|\mathbf{r}_t, \varphi) = -\frac{1}{2}\sum_{t=1}^{T}(\log(|R_t|) + \varepsilon_t' R_t^{-1}\varepsilon_t)$

Estimation of the volatility part ℓ_v is done by maximizing the sum of the individual GARCH likelihoods. Estimation of the correlation part is trickier because of the abstract states involved. Various solutions have been proposed.

In their seminal article, Fine, Singer, and Tishby (1998) used a *generalized Baum-Welch* algorithm based on an modified version of the *Inside-Outside* algorithm. This method requires to calculate all sequences that could be generated by each sub-model of each level of the hierarchy. With a computational complexity of $\mathcal{O}(NT^3)$, where N represents the number of hidden states and T the number of observations, this approach is suitable for small sequences but remains impractical for long sequences [9]. An alternative approach, first developed in Murphy and Paskin (2001) -and taking up again but with some modifications by Hung, Phung, and Venkatesh (2004) - provides an estimate by $\mathcal{O}(T)$. The idea is to consider the HHMM as a special case of Dynamic Bayesian Network (DBN). Note that a mechanic of a DBN is to repeat a chain of Bayesian network for each observation. The chains of the network are connected in time by a causal relationship which allow to track the evolution of the process. After converting the HHMM into a DBN, the estimation uses an classical Baum-Welch algorithm. However, if this is attractive from a computational point of view, it is still relatively complicated to implement. A third approach, suggested by Wierstra (2004), is to circumvent the problem posed by vertical transitions, which implies the existence of several paths to go from one state to another, by transforming the HHMM into a equivalent HMM. This *flattening* method, has the advantage of allowing to use the usual estimation methods of HMM with a complexity of $\mathcal{O}(T)$. Wierstra's tip is to redistribute vertical and horizontal probabilities such that no longer exists self-referential probabilities for internal states. This transformation allows to have a flat version of the HHMM in which there is only one path from a state to another. Xie (2005) succeeds to estimate the HHMM with HMM's standard tools while respecting the vertical dynamic of the model. In contrast to previous approaches, Xie's is build on a particular expression of the transition matrix. Instead of considering the whole dynamic of the model, the transition matrix is broken down into several sub-transition matrices for each hierarchical level. Each level of the tree is then linked with a transition matrix. This rewriting of the transition matrix allows to estimate the correlations with a version of the Baum-Welch algorithm very similar to that used in the context of standard HMM. The complexity of Xie's approach is of order $\mathcal{O}(DTN^{2N})$.

9. The inability of the Fine, Singer, and Tishby (1998)'s algorithm to deal with long sequence is not presented as a problem by the authors they apply their model for unsupervised handwriting recognition. In such applications, the size of the sequences are about fifty characters.

To estimate our HRSDC model, we follows a modified Xie's approach. The main advantage of its method is the speed of implementation. Xie builds the transition matrix by successive layers in order to have an expression of the likelihood of complete data that can be factorisable. In this framework, vertical transitions, which allows to activate the child of a parent state are given by :

$$\pi_q = \prod_{d=1}^{D} \pi^d_{q^d}, \ q = 0, ..., Q^D - 1 \tag{2.25}$$

The transition matrix is rewritten by layers, each layer represents a level of the hierarchy of the tree :

$$\tilde{a}^d(q', q) = \prod_{i=d}^{D} e^i_{q'^i} \pi^i_{q^i} \cdot a(q'^d, q^d) \tag{2.26}$$

This probabilities are called *hypertransition probabilities*. Aggregation of these probabilities leads to the *hypertransition matrix* of our model :

$$\tilde{A}(q', q) = \sum_{i=1}^{D} \tilde{a}^i(q', q) \tag{2.27}$$

Finally, the hypertransition matrix has size $(S \times S)$ where $S = card(\mathscr{S})$. This formulation is attractive in the sense that it enables working with a transition matrix similar to the classical HMM or Markov-Switching model. The hypertransition matrix of our HRSDC model is then written as :

$$\tilde{A} = \begin{bmatrix} a^2_{11} + e^2_1 \pi^2_1 a^1_{11} & a^2_{21} + e^2_2 \pi^2_1 a^1_{11} & e^2_3 \pi^2_1 a^1_{12} & e^2_4 \pi^2_1 a^1_{12} \\ a^2_{21} + e^2_1 \pi^2_2 a^1_{11} & a^2_{22} + e^2_2 \pi^2_2 a^1_{11} & e^2_3 \pi^2_2 a^1_{12} & e^2_4 \pi^2_2 a^1_{12} \\ e^2_1 \pi^2_3 a^1_{21} & e^2_2 \pi^2_3 a^1_{21} & a^2_{33} + e^2_3 \pi^2_3 a^1_{22} & a^2_{43} + e^2_4 \pi^2_3 a^1_{22} \\ e^2_1 \pi^2_4 a^1_{21} & e^2_2 \pi^2_4 a^1_{21} & a^2_{34} + e^2_3 \pi^2_4 a^1_{22} & a^2_{44} + e^2_4 \pi^2_4 a^1_{22} \end{bmatrix} \tag{2.28}$$

The hidden hierarchical structure requires twenty parameters. However, in practice, only sixteen parameters will be needed in order to carry out estimations due to constraints of stochastic matrices. In the next sub-section, we propose two ways to estimate the correlations. The first is based on the EM algorithm. The second is done with the Hamilton's filter.

Estimation by EM algorithm

To run EM algorithm [10], we need to write the quantity $\mathscr{Q}(\theta|\theta_k)$. With our hypertransition matrix, it is simply written as :

$$\begin{aligned} \mathscr{Q}(\theta|\theta_p) &= \mathbb{E}_{\theta_p}[\log f(\varepsilon_{1:T}, s_{1:T}; \theta)|\varepsilon_{1:T}] \\ &= \sum_{t=1}^{T} \sum_{i=1}^{S} \log(f(\varepsilon_t; \phi_i)) \mathbb{P}_{\theta_p}[s_t = i|\varepsilon_{1:T}] \\ &\quad + \sum_{t=1}^{T-1} \sum_{d=1}^{D} \sum_{i=1}^{S} \sum_{j=1}^{S} \log(\tilde{a}^d_{ij}) \mathbb{P}_{\theta_p}[s_{t+1} = j, s_t = i|\varepsilon_{1:T}] \\ &\quad + \sum_{i=1}^{S} \log(\nu_i) \mathbb{P}_{\theta_p}[s_1 = i|\varepsilon_{1:T}] \end{aligned} \tag{2.29}$$

To simplify the later formula 2.29, we define :

$$\begin{aligned} \gamma^p_t(q) &\overset{def}{=} \mathbb{P}_{\theta_p}[q_t = q|\varepsilon_{1:T}] \\ \xi^p_t(q', q, d) &\overset{def}{=} \mathbb{P}_{\theta_p}[q_t = q', q_{t+1} = q, e_t = d|\varepsilon_{1:T}] \end{aligned} \tag{2.30}$$

10. See appendix A.2 for a very brief review.

Re-estimation formulas are obtained by maximizing the expected value of the complete-data log-likelihood $\mathcal{Q}(\theta|\theta_p)$. In order to speed up this maximization, we can rewriting $\gamma_t^p(q)$ and $\xi_t^p(q',q,d)$ with the auxiliary variables defined by the Forward-Backward algorithm.

Forward-Backward algorithm The Forward-Backward algorithm is a useful method to reduce the calculus complexity [11]. It is to define two auxiliary quantities from which will be expressed reestimating formulas of the M step.

- Forward algorithm : the auxiliary quantity Forward, written $\alpha_t(q)$, is defined as the probability that the sequence $(\varepsilon_1,...,\varepsilon_t)$ to be in state q in the model θ :

$$\alpha_t(q) \overset{def}{=} \mathbb{P}_\theta[\varepsilon_{1:t}, q_t = q] \tag{2.31}$$

  Let $f^p(\varepsilon_t;\phi_q)$ the value of the likelihood at time t and iteration p. To simplify the notations, we will omit the superscript p and use the short-hand notation of $f_q(y_t)$. The two steps calculus for Forward quantity is as follow :

$$\begin{array}{l} ① \text{ Initialization} : \alpha_1(q) = \pi_q f_q(\varepsilon_1) \\ ② \text{ Iteration} : \alpha_{t+1}(q) = f_q(\varepsilon_{t+1}) \sum_{q'} \sum_d \alpha_t(q_t) \tilde{a}^d(q',q) \end{array} \tag{2.32}$$

  for $t = 1,...,T-1$ and $q = 0,...,Q^D-1$.
- Backward Algorithm : the Backward variable, written $\beta_t(q)$, is the inverse of the Forward. It is defined as the probability to generate the sequence $(\varepsilon_{t+1},...,\varepsilon_T)$ by leaving from state q in the model θ :

$$\beta_t(q) \overset{def}{=} \mathbb{P}_\theta[\varepsilon_{t+1:T}|q_t = q] \tag{2.33}$$

  We have then two steps to compute the Backward variable :

$$\begin{array}{l} ① \text{ Initialization} : \beta_T(q) = 1 \\ ② \text{ Iteration} : \beta_t(q) = \sum_{q_t} \sum_d \beta_{t+1}(q') f_{q'}(\varepsilon_{t+1}) \tilde{a}^d(q,q') \end{array} \tag{2.34}$$

  for $t = 1,...,T-1$ and $q = 0,...,Q^D-1$.

Auxiliary variables $\alpha_t(q)$ and $\beta_t(q)$ allows to rewriting the probabilities $\gamma_t^p(q)$ and $\xi_t^p(q',q,d)$ as follow [12] :

$$\begin{array}{l} \gamma_t^p(q) = \alpha_t(k)\beta_t(k) \\ \xi_t^p(q',q,d) = \alpha_t(q')\tilde{a}^d(q',q) f_{q'}(\varepsilon_{t+1})\beta_{t+1}(q) \end{array} \tag{2.35}$$

As these two variables are probabilities [13], they shall normalize as :

$$\begin{array}{l} \bullet \ \sum_k \gamma_t^p(q) = 1 \\ \bullet \ \sum_k \sum_{k'} \sum_d \xi_t^p(q',q,d) = 1 \\ \bullet \ \gamma_t^p(k') = \sum_k \sum_d \xi_t^p(k',k,d) \end{array} \tag{2.36}$$

Then, we can easily rewriting the reestimating formulas with $\gamma_t^p(q)$ and $\xi_t^p(q',q,d)$.

11. it lets you go of a complexity from $\mathcal{O}(e^T)$ to $\mathcal{O}(T)$ (see (Cappé, Moulines, and Ryden, 2005, chap. 3.2) and also (Baum and T., 1966, p. 170).

12. For a detailed calculation, see (Cappé, Moulines, and Ryden, 2005, chap. 3).

13. In practice, the calculation of auxiliary variables Forward and Backward revealed problems of *underflow*. It is therefore advisable to apply a scaling factor in order to facilitate the computation (see appendix A.3).

Re-estimation formulas Re-estimation formulas can be obtained by maximizing $\mathscr{Q}(\theta|\theta_p)$ subject to the stochastic constraints on the parameters (see sub-section 2.3). Then, we have only to differentiate with respect to the constraints and adding the appropriate Lagrange factors. Adopting Xie's notations for indexing the states, and writing $q = \overline{(rir')}$ and $q' = \overline{(rir'')}$ two states configuration which are identical up to the d^{th} level and have the same $(d-1)^{th}$ parent r such that $r = q^{1:d-1} = q'^{1:d-1}$, and $r' = q_t^{d+1:D}$ and $r'' = q_{t+1}^{d+1:D}$ two state configurations of a level below d, for each level $d \in \{1,...,\mathscr{D}\}$, the reestimating variables at iteration $p+1$ can be displayed as :

$$\begin{aligned}
&\bullet\ \hat{\pi}_r^d(j) = \frac{\sum_{t=1}^{T-1}\sum_{r'}\sum_{r''}\sum_i \xi_t^p(\overline{(rir')},\overline{(rjr'')},d)}{\sum_{t=1}^{T-1}\sum_{r'}\sum_{r''}\sum_j\sum_i \xi_t^p(\overline{(rir')},\overline{(rjr'')},d)} \\
&\bullet\ \hat{e}_r^d(i) = \frac{\sum_{t=1}^{T-1}\sum_{r'}\sum_{r''}\sum_{q'}\sum_{d'\leq d} \xi_t^p(\overline{(rir')},q',d')}{\sum_{t=1}^{T-1}\sum_{r'}\sum_{r''} \gamma_t^p((rir'))} \\
&\bullet\ \hat{a}_r^d(i,j) = \frac{\sum_{t=1}^{T-1}\sum_{r'}\sum_{r''} \xi_t^p(\overline{(rir')},\overline{(rjr'')},d)}{\sum_{t=1}^{T-1}\sum_{r'}\sum_{r''}\sum_j \xi_t^p(\overline{(rir')},\overline{(rjr'')},d)} \times (1-\hat{e}_r^d(i)) \\
&\bullet\ \hat{R}_q = \frac{\sum_{t=1}^{T} \varepsilon_t \cdot \gamma_t^p(q) \cdot \varepsilon_t'}{\sum_{t=1}^{T} \gamma_t^p(q)}
\end{aligned} \tag{2.37}$$

We simply use as stopping rule the conventional difference $\|\hat{\theta}^{p+1} - \hat{\theta}^p\| \leq 10^{-6}$.

Estimation Gradient methods

Using Gradient methods to estimate a HMM is made possible with the so-called Hamilton's filter (see Hamilton (1989, 1990, 1994)). This iterative filter allows one to make inference on the state of the unobserved Markov chain. With Hamilton's notations, let $\hat{\xi}_{t|t}$ a vector of size $(N \times 1)$ which elements $\xi_{jt} = \Pr[s_t = j|\mathscr{F}_{t-1},\theta]$, $j = 1,...,N$ are the conditional probabilities to be in regime j given the information set $\mathscr{F}_{t-1}$ at time $t-1$ and $\tilde{A}$ our hypertransition matrix of size $(N \times N)$ (see equation 2.27). Then one obtain a vector η_t of size $(N \times 1)$ of the elements are the densities under the N regimes, i.e. $f(\varepsilon_t|s_t = j,\mathscr{F}_{t-1};\theta)$, $j = 1,...,N$. Then, Hamilton shows that the observed density ε_t given the information set is written as :

$$f(\varepsilon_t|\mathscr{F}_{t-1};\theta) = \imath'(\hat{\xi}_{t|t-1} \odot \eta_t) \tag{2.38}$$

where $\imath'$ is a $(N \times 1)$ vector with all elements equal to 1. The conditional distribution of s_t can be deduced :

$$\begin{aligned}
\frac{f(\varepsilon_t, s_t = j|\mathscr{F}_{t-1};\theta)}{f(\varepsilon_t|\mathscr{F}_{t-1};\theta)} &= \mathbb{P}_\theta[s_t = j|\varepsilon_t,\mathscr{F}_{t-1}] \\
&= \mathbb{P}_\theta[s_t = j|\mathscr{F}_t]
\end{aligned} \tag{2.39}$$

Combining with equation 2.38, relation 2.39 has the following expression :

$$\mathbb{P}_\theta[s_t = j|\mathscr{F}_t] = \frac{f(\varepsilon_t, s_t = j|\mathscr{F}_{t-1};\theta)}{\imath'(\hat{\xi}_{t|t-1} \odot \eta_t)} \tag{2.40}$$

As the joint density $f(\varepsilon_t, s_t = j|\mathscr{F}_{t-1};\theta)$ is written as :

$$f(\varepsilon_t, s_t = j|\mathscr{F}_{t-1};\theta) = \mathbb{P}_\theta[s_t = j|\mathscr{F}_{t-1}] \times f(\varepsilon_t|s_t = j,\mathscr{F}_{t-1};\theta) \tag{2.41}$$

which is the density of the j^{th} regime. By collecting for the N regimes, the Hamilton's filter gives the following filtered probabilities :

$$\hat{\xi}_{t|t} = \frac{(\hat{\xi}_{t|t-1} \odot \eta_t)}{\imath'(\hat{\xi}_{t|t-1} \odot \eta_t)} \tag{2.42}$$

with forecasts calculated by :

$$\hat{\xi}_{t|t+1} = \tilde{A} \times \hat{\xi}_{t|t} \tag{2.43}$$

The inference for each date t is then found by iterating equations 3.31 and 2.43. For choosing starting value, we use one of the options proposed by (Hamilton, 1994, p. 693). He suggests to take the vector of unconditional probabilities π, which is the solution of the following system :

$$\begin{cases} \tilde{A}\pi = \pi \\ \imath'\pi = 1 \end{cases} \tag{2.44}$$

The starting value are then given by :

$$\pi = (M'M)^{-1}Me_{N+1} \text{with } M = \begin{bmatrix} \mathbf{I}_N - \tilde{A} \\ \imath' \end{bmatrix} \tag{2.45}$$

where e_{N+1} is the $(N+1)^{th}$ column of $\mathbf{I}_{N+1}$.

Smoothed probabilities

Hamilton's filter allows one to make inference about the state of the Markov chain at time t conditional on the information set up to time t. Kim (1994) has developed one filter in order to make inference with the whole information set. Instead of computing $\xi_{t|t}$, it permit to compute $\xi_{t|T}$ with $t < T$. The Kim's filter is computed with our hypertransition as in the classical case :

$$\hat{\xi}_{t|T} = \hat{\xi}_{t|t} \odot \{\tilde{A}'[\hat{\xi}_{t+1|T} \oslash \hat{\xi}_{t+1|t}]\} \tag{2.46}$$

where $\oslash$ denotes element-by-element division.

2.4 Applications

This section contains Monte-Carlo experiments and two applications based on real data. The first real database is that used by Colacito and Engle (2006) and contains daily data from S&P500 and 10-year bond futures (see appendix A.5). The second real database application is performed with that of Pelletier (2006). It contains the daily exchange rates at the close of the Pound, Deutschmark, Yen and Swiss-Franc against the US dollar (see appendix A.4).

2.4.1 Simulated data

In this sub-section, we compare the correlations estimates of our HRSDC model and the DCC_{ES} of Engle and Sheppard (2001) in a setting where the true correlation structure is known. For simplicity, it is done in a bivariate framework. We simulate six DGP. The first one has the following variance equations :

$$h_{1,t} = 0.01 + 0.05r_{1,t-1} + 0.85h_{1,t-1}$$
$$h_{1,t} = 0.12 + 0.1r_{1,t-1} + 0.85h_{1,t-1}$$

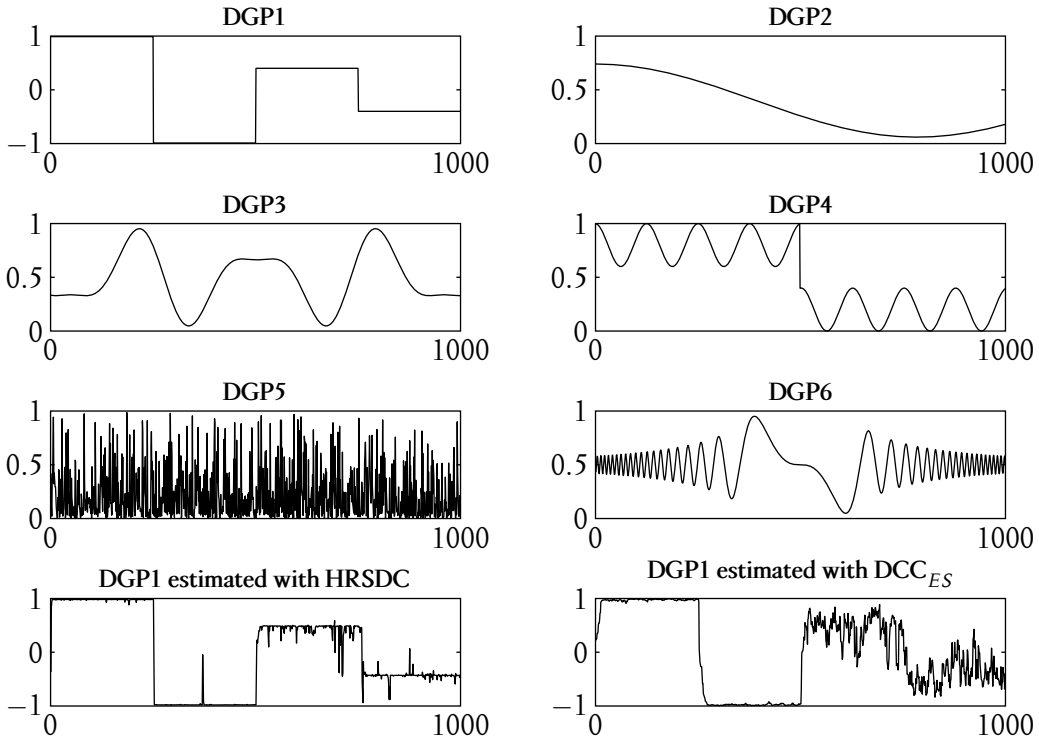


FIGURE 2.5 – Simulated Correlation Processes. The last two graphs are an example of the estimated correlations of the two models for the DGP1.

and the others (DGP2 :5) :

$$h_{1,t} = 0.01 + 0.04 r_{1,t-1} + 0.95 h_{1,t-1}$$
$$h_{1,t} = 0.01 + 0.2 r_{1,t-1} + 0.5 h_{1,t-1}$$

For the DGP1, the correlations process follow a TAR-CCC model with constant correlations. The DGP2 :5 are build with dynamic processes. The different DGP are labeled as :

- DGP1 : $R_{12,t} = \begin{cases} 0.99 \text{ if } t \in [1;250] \\ -0.99 \text{ if } t \in [251;500] \\ 0.4 \text{ if } t \in [501;750] \\ -0.4 \text{ if } t \in [751;1000] \end{cases}$
- DGP2 : $R_{12,t} = 0.4 + 0.34\cos(t/250)$
- DGP3 : $R_{12,t} = 0.5 + \cos(d/s) - (1/3)\cos(3d) + (1/7)\cos(5d)$, $d = (t-50)/145$, $s = 35$
- DGP4 : $R_{12,t} = \begin{cases} .8 + .2\cos(t/20) \text{ if } t \in [1;500] \\ 0.2 + .2\cos(t/20) \text{ if } t \in [501;1000] \end{cases}$
- DGP5 : $R_{12,t} = 0.99 - \frac{1.98}{1+\exp(0.5\max(\varepsilon^2_{1,t-1},\varepsilon^2_{2,t-1}))}$
- DGP6 : $R_{12,t} = 0.5 + \sin(s^3)/(1+\sqrt{|s^3|})$, $s = 5 - t/100$

The simulated correlations are voluntarily pathological in order to test the accuracy of the correlations estimates corresponding of very volatile/distress periods of financial markets. The DGP2 :4 ;6 are each built with sinusoidal functions to create correlations with different regimes or sub-regimes. The DGP5 is used by Long and Ullah (2005) and corresponds of the stylized fact pointed out by Longin and Longin and Solnik (2001) that correlations among assets tend to increase during volatile periods.

	DGP1	DGP2	DGP3	DGP4	DGP5	DGP6
MAE$_1$						
DCC	.1331	.5125	.3774	.4237	.5298	.2764
HRSDC	.0562	.5521	.3836	.4642	.5386	.2831
MAE$_2$						
DCC	.0553	.2901	.1700	.2361	.2824	.0778
HRSDC	.0235	.3408	.1660	.2659	.3008	.0835
DQ 5% *(strategy EW)*						
DCC	.7594	.5884	.6499	.9299	.5784	.9586
HRSDC	.5966	.6985	.7814	.9699	.4263	.9800
DQ 1% *(strategy EW)*						
DCC	.3359	.9944	.9970	$1e^{-4}$	.0217	.9925
HRSDC	.4382	.9733	.9995	$2e^{-4}$	.0215	.9929
DQ 5% *(strategy LS)*						
DCC	.7969	.1616	.5136	.9532	.7284	.7068
HRSDC	.8731	.0912	.3509	.9216	.6574	.7321
DQ 1% *(strategy LS)*						
DCC	.4294	$4.3e^{-5}$	.9986	.9996	.01	.0523
HRSDC	.3896	.2685	.9959	.9994	.01	.0136

TABLE 2.1 – Performance measures results.

The performance measures we use are very similar of theses used by Engle (2002). We first calculate two versions of a very classical loss function, which are computed as follow :

$$MAE_1 = \frac{1}{T}\sum_{t=1}^{T}\left|\hat{R}_t - R_t\right|$$
$$MAE_2 = \frac{1}{T}\sum_{t=1}^{T}(\hat{R}_t - R_t)^2$$

For the second type of measure, we follow the methodology of Engle (2002) by considering the loss function of the Value-at-Risk (VaR). Recall that for a portfolio with a share w invested in the in the first asset and $(1-w)$ in the second, the VaR assuming normality can be computed as :

$$VaR_t^{\alpha} = \Phi_t^{-1}(\alpha)\sqrt{(w^2\hat{H}_{11,t} + (1-w)^2\hat{H}_{22,t} + 2w(1-w)\hat{R}_{12,t}\sqrt{\hat{H}_{11,t}\hat{H}_{22,t}})}$$

The loss function of the VaR is then defined by :

$$hit_t = \mathbb{1}_{\{wr_{1,t}+(1-w)r_{2,t}<-VaR_t\}} - \alpha$$

We then use the *in-sample Dynamic Quantile* (DQ) test introduced by Engle and Manganelli (2004). This consists to test if all the coefficients of the regression of the violation process hit_t with its lagged values and others exogenous variables are equals to zero. To compute this F test, we use as explanatory variables five lags hit_t and the current value of the VaR. We test two numbers of rejections (1% and 5%) and two portfolios : an equal-weighted ($w = 0.5$, strategy EW) and a long-short (1 and -1, strategy LS). Table 2.1 presents the results of the performance measures.

The results show that the HRSDC performs better than the DCC when correlations are piecewise constant. This result is consistent with what could be expected since the HRSDC varied correlations between several constant correlations matrices over time. Once the simulated correlations are no longer constant within a specific sample, the DCC model has better MAE, which, once again, seems normal. However, differences in the MAE between the DCC and the HRSDC are

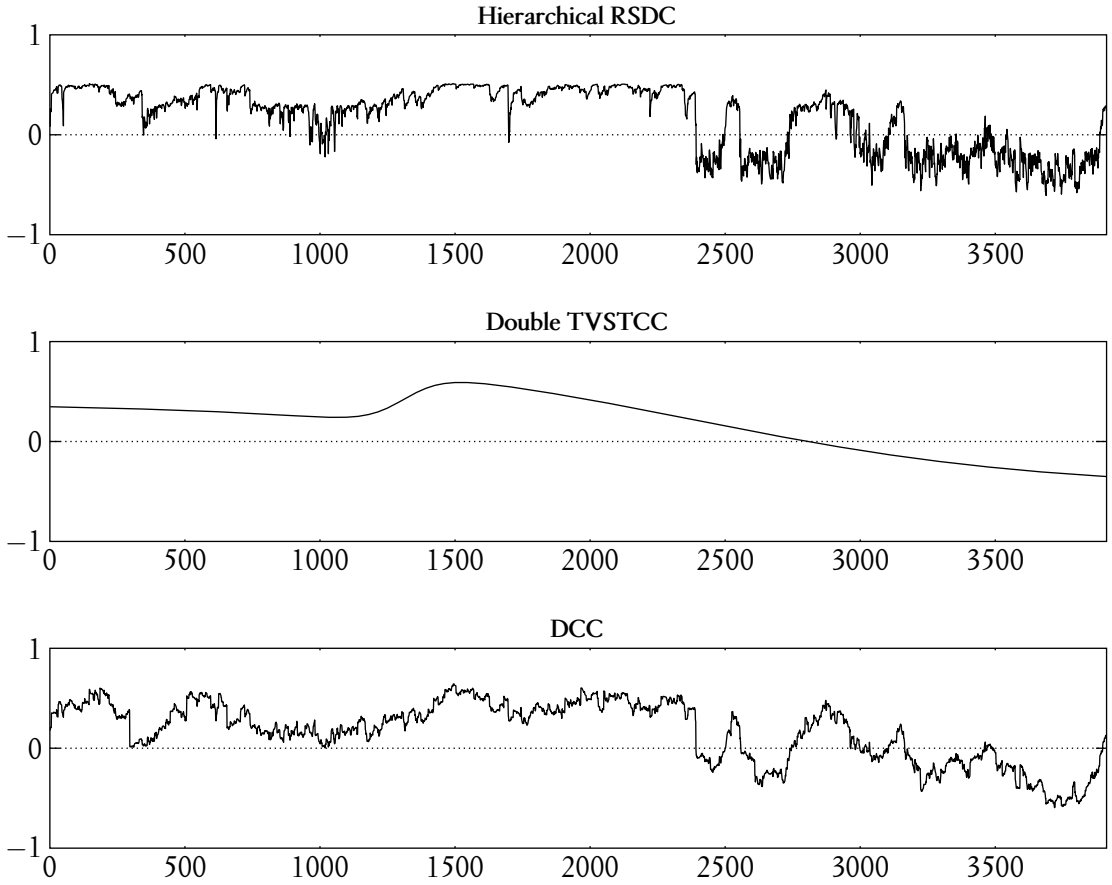


FIGURE 2.6 – Estimated correlations of the HRSDC, DCC_{ES} and DSTCC with the sample of Colacito and Engle (2006).

comparatively small. The dynamic process resulting from a combination of regimes of the HRSDC does not appear to fare too badly in comparison to the autoregressive dynamic of the DCC.

The results of the DQ test are more mixed and appear highly dependents on the strategy chosen. The DCC is the best for EW strategy, while HRSDC dominates on the LS strategy. A simple addition of the best values shows that the HRSDC seems preferable to DCC. Nevertheless, it would be unfair to conclude to the dominance of HRSDC under the DCC. It is just better to point out that the HRSDC remains credible face to the DCC.

Finally, the HRSDC remains a tool of great efficiency when correlations are piecewise constant (DGP1), but becomes less effective when it is no longer the case (DGP2 :6). It seems that the autoregressive dynamic of the DCC remains very efficient when correlations oscillate around a trend value. However, the appeal of the HRSDC lies in its explanatory power. Apart from providing a simple measure of correlations, its structure allows one to bring out the components of their overall variability by increasing the granularity of regimes.

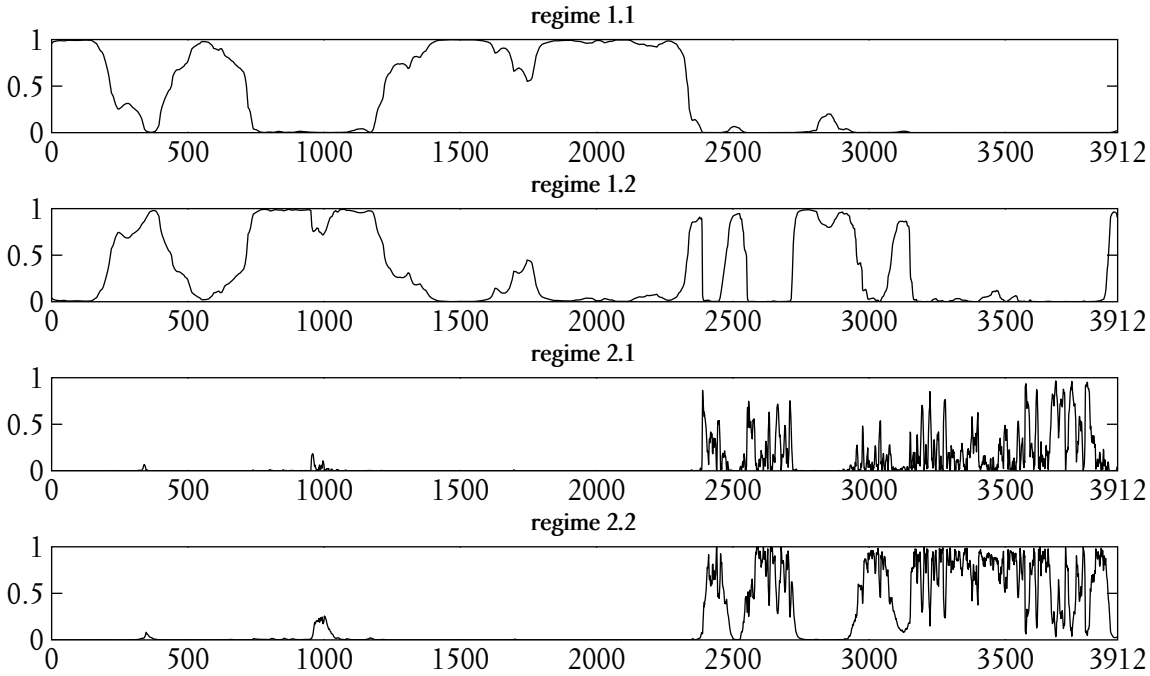


FIGURE 2.7 – Smoothed probabilities of the HRSDC. Regimes 1.1 and 1.2 correspond to the emitting states $\{s_1^2, s_2^2\}$ and regimes 2.1 and 2.2 to $\{s_3^2, s_4^2\}$.

2.4.2 Correlations between S&P500 index futures and 10-year bond futures

The first application with a real database is based on the bivariate sample of Colacito and Engle (2006). It contains daily returns of S&P500 index futures and 10-year bond futures from January 1990 to August 2003 (see appendix A.5). This data are also used by Silvennoinen and Teräsvirta (2007) in demonstrating to apply their DSTCC model. The individuals volatilities are obtained by running a GARCH(1,1) model. We later obtain the correlations by running three different models in order to make comparisons. These models are HRSDC, DCC_{ES} and DSTCC. For this sample, the correlations of the DCC_{ES} and DSTCC are estimated using gradient methods while the correlations of the HRSDC are computed with EM algorithm. The figure 2.6 shows the estimated correlations for the three models.

In our application, transition variables for the DSTCC are defined as $m_{1t} = m_{2t} = t/T$ (calendar time). In their study, Silvennoinen and Teräsvirta (2007) used another specification : one of the transition variable is calendar time while the other is VIX index [14]. This particular approach is motivated by investigate the fact that correlations among S&P500 and bond futures increase with distress of the market. As our model can not incorporate exogenous variable, calendar time allows to the DSTCC to be similar to the HRSDC. They are together conditional on time.
Results of the estimated parameters of the HRSDC are presented in appendix A.6 and the smoothed probabilities corresponding to the secondary regimes are in the figure 2.7. The model has correctly identified two sub-regimes with positive correlations and two sub-regimes with negative correlations. Adding the smoothed probabilities of each pairs of sub-regimes give the smoo-

14. The Vix index (for Chicago Board Options Exchange volatility index) is build with of selected basket of the implied volatility of S&P500 index options. It gives a measure of expectation of the market's volatility over the next 30 days. The VIX index can be freely downloaded at : `http://www.cboe.com/micro/vix/introduction.aspx`

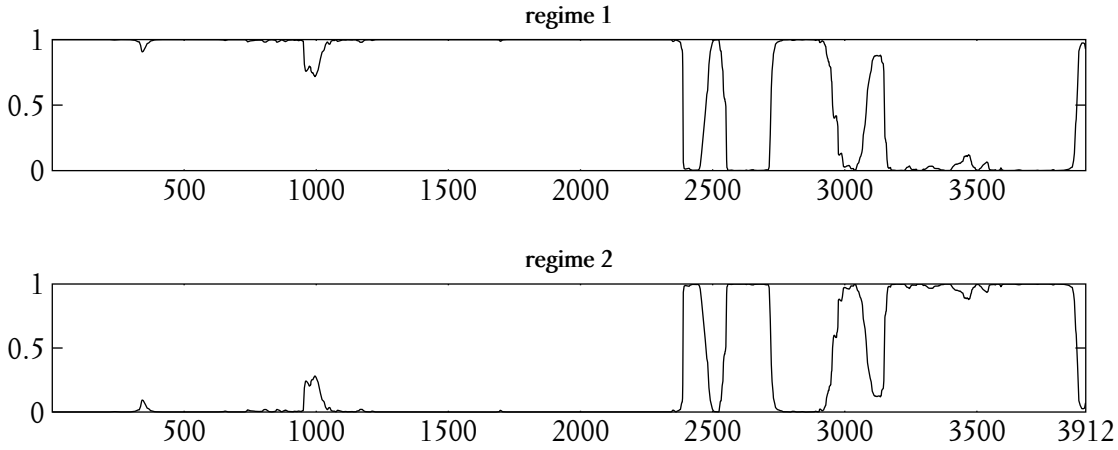


FIGURE 2.8 – Smooth probabilities of the regimes defined by the internal states i_1^1 and i_2^1 (first application).

thed probabilities that we would have had with Markov-Switching model with two regimes. This is shown on the Figure 2.8. The results for the DSTCC (with the calendar time specification) show that it does not accurately captures the *punctual* changes in regimes. Correlations estimate have the form of a curve. Rather than a local measure, this model serves to indicate a trend. For example, in the DCC and HRSDC models, correlations go to a negative correlations regime around the 2400^{th} observation whereas they are still positives in the DSTCC. These is due to the fact that the DSTCC has only two transition functions. A solution to this problem could be to introduce more transition functions as in Amado and Teräsvirta (2008). If the DCC and HRSDC correlations estimate seem similar, the advantage of HRSDC in the case study is to explain nuances in the dynamics through the decomposition in sub-regimes. The increasing in granularity ends up in a finer definition of the transition.

2.4.3 Correlations between exchange rate data

In this second application, we apply our HRSDC model to the sample used by Pelletier (2006). These series are exchange rate data and are plotted on the figure A.2 (see also appendix A.4). As before, the dynamic of the standard deviations is obtained with a GARCH(1,1) model. For this case study, only DSTCC and HRSDC models are considered. We use the two-step maximum likelihood estimation using gradient methods for both models. Figure 2.9 shows plots of the estimated correlations for the two models. Parameters estimated for the HRSDC are in the table A.2 (see appendix A.6). Smooth probabilities for the HRSDC can be seen in the figure 2.10.

As we see from figure 2.9, as in the previous application, the DSTCC (with its calendar time specification, i.e. $m_{1t} = m_{2t} = t/T$) allows one at best to perceive a trend on the evolution of correlations in the sample. The reason for this limitation is often that, with only two transition functions, the DSTCC can not capture more than three regimes. The direct consequence of this shortcoming is that the DSTCC fails to capture the time specific extreme regimes, as it shows by example on the Swiss Fr/Deutschmark correlations. These are indeed marked by two peaks of correlations close to zero as they oscillate the rest of the time around a value about 0.8.

In this sample, the HRSDC clearly identifies four sub-regimes. Recall that the first primary regime associated with the internal state i_1^1 is a combination of the secondaries regimes depending of the two emitting states s_1^2 and s_2^2. Increasing the granularity provided by the hidden hierarchical

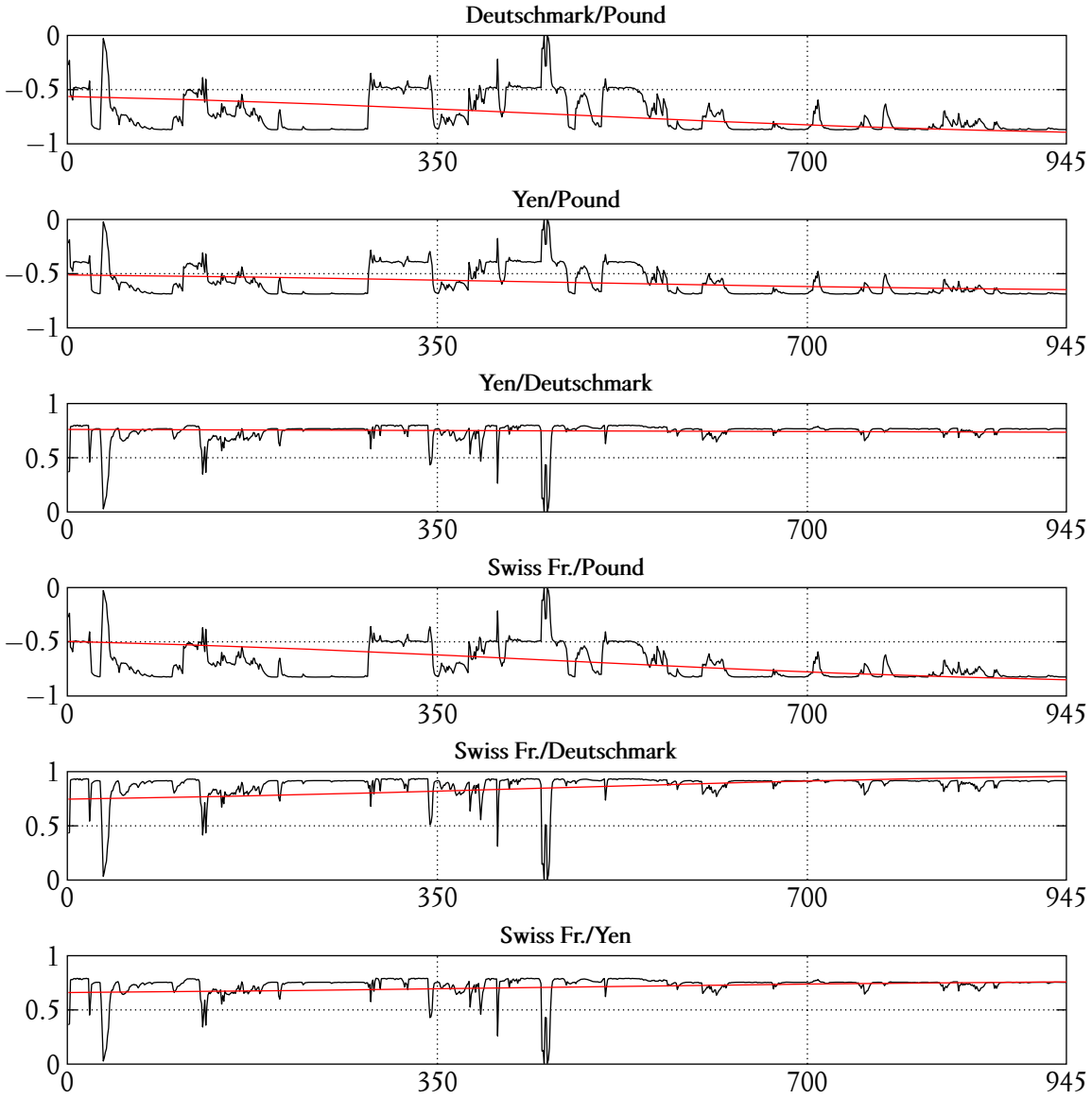


FIGURE 2.9 – Estimated correlations for the HRSDC (in black) and DSTCC (in red).

structure allows to highlight the existence of a sub-regime linked to s_2^2. This sub-regime occurs only very rarely within the sample, around the 450^{th} observation. But it permit one to capture an extreme and punctual behavior of the correlations process. This element of the global dynamic of correlations would be go unnoticed with a classical Markov-Switching model (because that model is too limited and not significant enough in relation to the size of the sample).

This application with a sample of four series brings to light a problem in the correlations specification. As we have said before, estimated parameters have been obtained using Gradient methods to estimate the HRSDC. This choice is not fortuitous. In fact, our experiments shows that the non constraint specification defined by equation (2.17) meets with difficulties in returning the maximum of the objective function with the use of EM algorithm. The cause of the problem is to be found in the constraint Choleski representation used for the correlation matrix. Without this

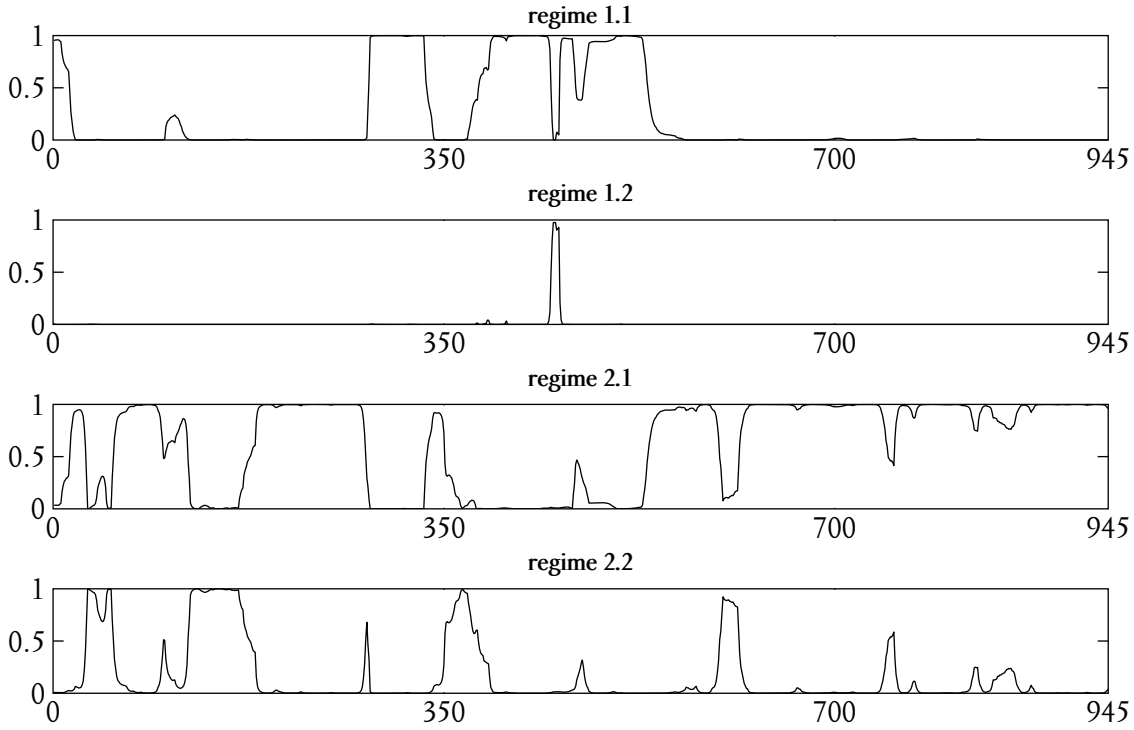


FIGURE 2.10 – Smoothed probabilities for the HRSDC (exchange rate data).

transformation, the derivatives of the objective function produce a form of the correlation matrix with no guarantees to have ones in the diagonal and PSD. But, the substitute of the Choleski representation for the diagonal elements :

$$c_{j,j} = \sqrt{r_{i,i} - \sum_{i=1}^{j-1} c_{j,i}^2} \tag{2.47}$$

by its constraint version :

$$c_{j,j} = \sqrt{1 - \sum_{i=1}^{j-1} c_{j,i}^2} \tag{2.48}$$

strongly disrupts the convergence of the EM algorithm. That's why it is preferable to use the specification defined by equation (2.19) which can be estimated by gradient methods using Hamilton's filter. In this example, despite of a huge number of iterations, the EM algorithm has not converged whereas Gradient methods rapidly reach the optimum of the objective function. Nevertheless, even the constraint specification (equation (2.19)) would need numerous parameters and the estimation of large correlation matrix could turn out to be cumbersome.

2.5 Conclusion

In this paper, we have presented a new multivariate GARCH with dynamic correlations. This extension, called Hierarchical RSDC (HRSDC), can be view as a special case the RSDC model of Pelletier (2006). The HRSDC is a Markov-Switching class model with a correlation process

bounded by four correlation matrices constant over time. Our model is innovative in that it is built on a hierarchical hidden structure introduced by Fine, Singer, and Tishby (1998).
The main advantage of this hidden tree-like structure is to increase the granularity of the regime. This make it possible to define different types of regime ; in our case, primary and secondary regime. Applied to correlations modeling, the HRSDC allows one to capture thinner nuances than is possible with the classical Markov-Switching approach. Monte Carlo experiments and applications on real data show that this approach improves understanding of the dynamic of the correlations. The application of the HRSDC to estimate the correlations between S&P500 index futures and 10-year bond futures or between exchange rate data has brought to light the existence of sub-regimes which other regime switching models have so far been unable to do.
While the results in this paper show that the HRSDC has a good explanatory power, we may be aware that it has several limitations as well. The first concerns model selection. In this study, our model is built from a symmetric hidden tree, with two primaries regimes, each of them with which one sub-HMM each. It is based on a very simple structure. It should be possible to build an asymmetric tree with much levels of depth. However, finding the best hierarchy, i.e. split/merge/swap levels, could be an open problem. Xie (2005) use the Reversible-Jump MCMC (RJMCMC) of Green (1995) for model selection. While this approach may appear attractive, one will have to contend with the questions of size would have to be tackled. The second problem relates to the specification of the correlation matrix. The specification of Pelletier is not suitable in the modeling of large correlation matrix. As the hierarchical hidden structure model is a plug-in method, finding a specification for large correlation matrices would be a valuable avenue for future research.

CHAPITRE 3

A Markov-switching Dynamic Conditional Correlation Model with Factorial Hidden Markov Representation.

3.1 Introduction

During the past few years, financial econometrics has shown a wide range of new developments in the modeling of variances and correlations of financial series. In this new field, the work of Bollerslev (1990), which introduced a multivariate GARCH with constant conditional correlations (the well-known CCC), has brought to conditional correlation models an enormous amount of attention. Since empirical studies (see Longin and Solnik (1995, 1996, 2001) among others) have pointed out the fact of fluctuations in correlations among series according to the degree of the volatility of financial markets, the focus has now shifted to on dynamic correlation models. Engle and Sheppard (2001) (and also around the same time Tse and Tsui (2002)) proposed a generalization of the CCC by making correlations time-varying (the DCC model). This seminal approach has been the starting point of many extensions which allow for richer correlation dynamics while making estimation feasible [1].

Since Hamilton (1989), Markov-Switching has been extensively used in econometrics in the field of time series. The first applications of Markov-Switching to ARCH models came from Cai (1994) and Hamilton and Susmel (1994). Gray (1996) solved the problem of path-dependency and proposed an MS-GARCH by taking the conditional variance only on the current regime by the expectation of the lagged variance. This specification was further improved upon by Klaassen (2002) with the introduction of a broader information set. Finally, Haas, Mittnik, and Paollela (2004b) generalized the previous approaches by making the GARCH equations corresponding to hidden states evolving in parallel. Extensions to the multivariate case have been proposed by Pelletier (2006), Billio and Caporin (2005) and also Haas and Mittnik (2008) [2].

In this paper, we propose an extension of the Markov-Switching version of the DCC based on the approach of Haas and Mittnik (2008). Indeed, one limitation of the previous models is that all elements of the correlation matrix must switch together. In the case of the model of Pelletier (2006) with two states, correlations evolve between two extreme constant correlations. In Billio and Caporin (2005), and also in Haas and Mittnik (2008), correlations are defined by two non-linear processes. When a switch occurs, all elements of the correlation matrix switch simultaneously. We argue that this condition may be too restrictive. The main innovation of our proposed approach is that it allows each correlation to have its own switching dynamic. Thus, elements of the correlations can jump to different times, while, in the classical models, all elements switch at the same time.

This constraint of a common switch for all the elements of the correlation matrix is relaxed by introducing a Markov switch following the factorial representation developed by Ghahramani and Jordan (1997). A factorial representation should not be confused with the factor models usually used in econometrics. The factorial hidden Markov model (FHM) can be viewed as a generalization of the classical hidden Markov model in which hidden states are factored into multiple hidden state variables. Each element of the correlation matrix follows its own Markov chain so that the correlations between the first and the second time series can have a different switching dynamic from correlations between the first and the third time series.

This paper is organized as follows. The model is defined section 3.2. First, we briefly recall the basics of the DCC of Engle and Sheppard (2001) and then outline our model. In section 3.3 the model is applied to data examples. Section 3.4 contains the concluding remarks and an outline of proposed areas for further research.

1. see Bauwens, Laurent, and Rombouts (2006) or Silvennoinen and Teräsvirta (2009) for a complete survey.
2. These models will be reviewed in the next section.

3.2 The Model

In this section we first define the basic framework of dynamic correlation models and then explain the FHM approach.

3.2.1 Conditional Correlation Models

Starting with Bollerslev (1990), the basic framework for conditional correlation models defines the conditional variance-covariance matrix of returns r_t of size $K \times 1$ expressed as follows :

$$H_t = D_t R D_t \tag{3.1}$$

where R is a $K \times K$ constant correlation matrix and D_t a $K \times K$ diagonal matrix containing univariate time varying standard deviations such that :

$$D_t = \text{diag}\{h_{i,t}^{1/2}\},\ i = 1, ..., K \tag{3.2}$$

These elements can be extracted by applying a GARCH(p,q) for each series $i = 1, ..., K$:

$$h_{i,t} = \omega_i + \sum_{j=1}^{q} \alpha_{i,j} r_{i,t-j}^2 + \sum_{j=1}^{p} \beta_{i,j} h_{i,t-j} \tag{3.3}$$

with usual stationarity and non-negativity restrictions. Another way of looking at this relation becomes available if we rewrite equation (4.2.2) as follow. Let ε_t the standardized residuals expressed as :

$$\varepsilon_t = D_t^{-1} \mathbf{r}_t \tag{3.4}$$

The conditional correlations are then simply the expectation of the standardized residuals :

$$\mathbb{E}_{t-1}[\varepsilon_t \varepsilon_t'] = D_t^{-1} H_t D_t^{-1} = R \tag{3.5}$$

Therefore, despite its computational attractiveness, the assumption of constant correlations seems to be too parsimonious and seriously unrealistic in empirical applications (see Longin and Solnik (1995, 1996, 2001) among others). Engle and Sheppard (2001) extended this previous approach by making the correlations to be time-varying :

$$H_t = D_t R_t D_t \tag{3.6}$$

This model (called DCC) first defines a dynamic for the covariance of the residuals from which conditional correlations are computed. The conditional covariance dynamics of the standardized residuals V_t follow a BEKK-like process :

$$Q_t = (1 - a - b)\overline{Q} + a\varepsilon_{t-1}\varepsilon_{t-1}' + bQ_{t-1} \tag{3.7}$$

where $\overline{Q}$ is the unconditional covariance matrix of the standardized residuals and a and b are two positive scalars such that $a + b < 1$. Under these restrictions the matrix V_t is positive definite and the conditional correlation matrix is then obtained by :

$$R_t = \text{diag}\{Q_t\}^{-1/2} Q_t \text{diag}\{Q_t\}^{-1/2} \tag{3.8}$$

One interesting feature of this approach is that it involves only two parameters to define covariance dynamics and a such specification added advantage of being parsimonious in the field of multivariate GARCH models. This original parametrization opened up a new field in multivariate variance modeling and dynamic correlation models have now become a new center of attention. We no longer outline these extensions and we refer to Silvennoinen and Teräsvirta (2009) and Bauwens, Laurent, and Rombouts (2006) for an updated review of these approaches.

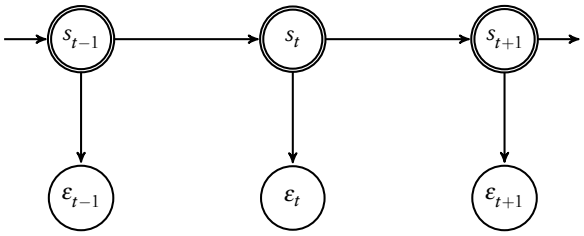


FIGURE 3.1 – The Markov-Switching model viewed as a dynamic Bayesian network.

3.2.2 The main approach

Our approach extends the Markov-Switching model to modeling dynamic correlations. We shall briefly recall the main idea of the Markov-Switching model. This approach assumes that the observed variable ε_t has a specific relation with a hidden state s_t which follow a first-order Markov chain. Each observation is linked to a hidden state by a probability, and the switch from one state to another is specified by a transition matrix. This linking relation is generally a first-order Markov chain, that means the state s_t is independent of state lagged state except s_{t-1} :

$$\mathbb{P}[s_{t+1} = i | s_t = j] = p_{ij} \tag{3.9}$$

These probabilities are grouped together to make an $N \times N$ transition matrix :

$$\mathrm{P} = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{bmatrix} \tag{3.10}$$

where $\sum_{j=1}^{N} a_{ij} = 1$ for each $i = 1, ..., N$. In its more general representation, any Markov-Switching model can be written as :

$$\varepsilon_t | \mathscr{F}_{t-1} \sim f(\varepsilon_t; \theta) \tag{3.11}$$

$\mathscr{F}_{t-1}$ denotes the information set at time $t-1$, and θ the parameter set. The density function $f(\varepsilon_t; \theta)$ given past observations over N states of the Markov chain can then be expressed as :

$$f(\varepsilon_t; \theta) = \sum_{n=1}^{N} f(\varepsilon_t | s_t = n; \theta_n) \mathbb{P}[s_t = n; \theta_n] \tag{3.12}$$

with θ_n denoting the parameters of each regime n for $n = 1, ..., N$. The main idea here is to allow the parameters to switch according to the value of the hidden Markov chain at each point of time. This can be expressed graphically by a dynamic Bayesian network (DBN) (see figure 3.1). The idea of introducing Markov-Switching in dynamic correlations models has been motivated by severals empirical applications. In particular, Longin and Solnik (2001) has shown that correlations tend to increase when volatility is high and Ang and Chen (2002) and Campbell, Koedijk, and Kofman (2002) found a significantly increased correlation in international equity returns in bear markets. In this way, Pelletier (2006) suggested a specification where correlations switch between various constant correlations matrix :

$$R_t = \sum_{n=1}^{N} \mathbb{1}_{\{s_n = i\}} R_n \tag{3.13}$$

where R_n is the constant correlation matrix of regime n, and $\{s_t\}_{t \in \mathbb{N}}$ a first-order Markov chain. Because correlation evolves between constant correlations matrices, this model can be viewed as

a hybrid approach between the CCC and the DCC. But despite its simple specification, this representation involves a large number of parameters. To avoid this problem, Pelletier (2006) proposed a constraint formulation :

$$R_t = \Gamma \lambda_{s_t} + \mathbf{I}_K \left(1 - \lambda_{s_t}\right) \tag{3.14}$$

with $\lambda_{s_t} \in (0,1)$ a regime-dependent weighting variable, Γ a constant correlation matrix, and $\mathbf{I}_K$ a $K \times K$ identity matrix. Nevertheless, this formulation does not permit the correlations to change signs. In order to bypass these drawbacks, Billio and Caporin (2005) introduced a regime switch in equation (3.7) such as :

$$Q_t = \left(1 - a_{s_t} - b_{s_t}\right)\overline{Q}_{s_t} + a_{s_t}\varepsilon_{t-1}\varepsilon'_{t-1} + b_{s_t}Q_{t-1} \tag{3.15}$$

where a_{s_t} and b_{s_t} are two regime-dependent scalars, and $\overline{Q}_{s_t}$ an unconditional regime-dependent covariance matrix of the standardized residuals. The restrictions to have a positive definite correlation matrix is to have $a_{s_t} + b_{s_t} < 1$ and $\overline{Q}_{s_t}$ definite positive. However, a clear drawback of this approach is that it requires approximations to be estimated, and this leads to difficulties in the interpretation of the covariance process for each regime.

Haas and Mittnik (2008) have developed a multivariate Markov-Switching GARCH model based on the assumption of Haas, Mittnik, and Paollela (2004b). This approach is applied on a full BEKK formulation but can be easily extended to the DCC as the dynamic of the residuals in the follows a BEKK process. Haas and Mittnik (2008) suggest that the conditional covariance process has K independent covariance processes. It can be written as follows :

$$Q_{t,s_t} = C_{s_t} + A_{s_t}\varepsilon_{t-1}\varepsilon'_{t-1}A'_{s_t} + B_{s_t}Q_{s_t,t-1}B'_{s_t} \tag{3.16}$$

where A_{s_t} and B_{s_t} are symmetric matrices and C_{s_t} is a positive definite matrix. This specification does not require approximations as the conditional covariance matrix is written as a function of the current state s_t and the squared standardized residuals. This specification has several advantages. As pointed out by Haas and Mittnik (2008), it seems to be the *most natural* approach for multi-regime GARCH models since the K covariance processes evolve independently, making the task of identifying the low- and high-volatility periods easier. The possibility this model offers of applying maximum likelihood methods to estimations enhances its appeal from a practical point of view.

3.2.3 Definition of the Model

Despite its high capabilities in modeling regime switches, we argue in this paper that the classical Markov-Switching approach may in certain situations suffer from a lack of flexibility. Indeed, switches between regimes then impose strong restrictions on the parameter set. In the case of equation (3.16), the switch from one regime to another assumes that all the conditional covariances switch together from one regime to another. Further, in that approach, all the covariances are assumed to have the same switching date. That precludes the possibility of differentiating the switches among individual covariances within the covariance matrix. This is a restriction that needs to be relaxed.

The model we have proposed holds particular relevance in economics, such as in a study of the linkages in the correlations of exchange rates. If one were to use the standard DCC model, one would find oneself restricted with the dynamic of correlations being constant for the duration of the given period. However, such a view may not be realistic during periods of financial turbulence, and a DCC based on the MMS-GARCH structure of Haas and Mittnik (2008) then may be the proper specification to use in modelling possible changes in the correlations. At the same time, it must

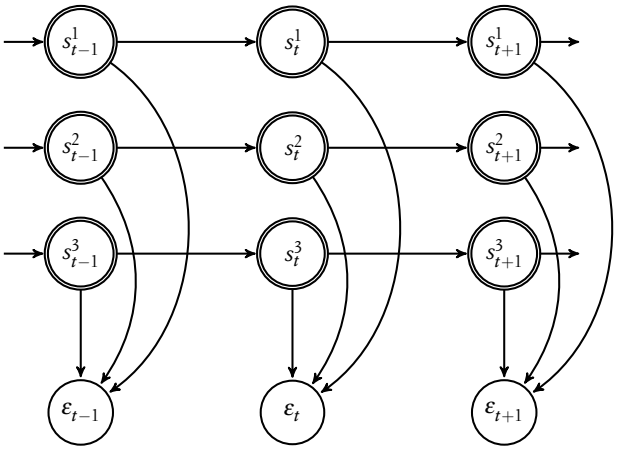


FIGURE 3.2 – FHMM viewed as DBN.

be recognized that in that case all the correlations are assumed to switch at the same time. While the specification provides considerable explanatory power, and enables a clear identification of crisis/non-crisis periods, this assumption may prove to be too sweeping, if not too simplistic. The model we have proposed assumes dynamic conditions for individual switching for each element of the correlation matrix, and thus overcomes the shortcomings of the models described.
Our model is based on the factorial hidden Markov model (FHMM) where each covariance of the conditional covariance matrix has its own dynamic. This extension of the hidden Markov model, developed by Ghahramani and Jordan (1997), generalized the previous Markov-Switching approach by representing the hidden state in a factored form. Each hidden state is factored into multiple hidden state variables evolving in parallel. Each chain has dynamics similar to those of a basic hidden Markov model. We shall consider in this paper that the chains are independent and we shall make the usuals assumptions of aperiodicity, irreducibility and ergodicity. However, our model will differ from the hidden Markov model in that the probability that an observation will occur at each point of the time segment will depend upon the current state in all of the chains. Representation as dynamic Bayesian networks is shown in figure 3.2. Comparing to equation (3.12), the density function $f(\varepsilon_t;\theta)$ given past observations over the set of M parallel Markov chain $s^1,...,s^M$ with N states each can then be expressed as :

$$\mathbb{P}[s^1_{t+1}=i,...,s^1_{t+1}=i|s^1_t=j,...,s^1_t=j]=\prod_{m=1}^{M}\mathbb{P}[s^m_{t+1}=i|s^m_t=j] \tag{3.17}$$

Since the state space made up of the cross product of these state variables, this representation can model several processes with their own independent dynamics. It thus becomes possible to build a Markov-Switching setup for equation (3.16) where covariances are linked each to its own Markov chain so that it allows allow them to switch independently of one another. This statement can be formally expressed as follows :

$$s_t=(s^1_t,...,s^M_t) \tag{3.18}$$

The expression for the conditional covariance process of the standardized residuals can then be written as :

$$Q_{t,s_t}=C_{s_t}+A_{s_t}\varepsilon_{t-1}\varepsilon'_{t-1}A'_{s_t}+B_{s_t}Q_{t-1,s_t}B'_{s_t} \tag{3.19}$$

Each Markov chain s^i_t is linked to a parameter and has its own transition matrix, respectively $\mathrm{P^i}$ with $i=1,...,M$. For convenience, we will assume that all Markov chains have N states each. The resulting state space is then based on N^M combinations.

So as to avoid an excessively large number of parameters, while at the same time ensuring the quality of estimation, we shall choose diagonal matrix for A_{s_t} and B_{s_t} such that :

$$A_{s_t} = \text{diag}(a_{s_t^1}, a_{s_t^2}, ..., a_{s_t^M}) \tag{3.20}$$

and similarly :

$$B_{s_t} = \text{diag}(b_{s_t^1}, b_{s_t^2}, ..., b_{s_t^M}) \tag{3.21}$$

Then, the ij^{th} element of V_{t,s_t} will have the following expression :

$$Q_{ij,t,s_t} = c_{ij,s_t} + a_{i,s_t^m} a_{i,s_t^m} \varepsilon_{t-1} \varepsilon'_{t-1} + b_{i,s_t^m} b_{i,s_t^m} Q_{ij,t-1,s_t} \tag{3.22}$$

for $m = 1, ..., M$ and $i, j = 1, ..., K$.

The stationarity condition yet assumes that Q_{t,s_t} is positive definite and :

$$a_{i,s_t^m} a_{i,s_t^m} + b_{i,s_t^m} b_{i,s_t^m} < 1 \tag{3.23}$$

with $m = 1, ..., M$ and $i = 1, ...K$. It also requires the matrix C_{s_t} to be symmetric and positive definite and the initial covariance matrix to be positive definite. Given this specification, our model does not suffer from the problem of path dependency. Built on the basic concept of Haas and Mittnik (2008) and can be view as a special case of their approach. While Haas and Mittnik (2008) assume a covariance matrix Q_t defined by K separate BEKK models, where all the element switch together from one regime to another, our model assumes that each element of the matrix Q_t evolve independently from each other in their switching dynamic.

3.2.4 Estimation via equivalent HMM representation

As is well known, the Holy Grail in multivariate GARCH modelling has been the endeavour to reconcile two apparently contradictory objectives : having a model possessing good explanatory power, and one which at the same time requires only a small number of parameters to make the process of estimation feasible. This challenge has prompted the choice of a scalar BEKK representation for our covariance process. Nevertheless, our specification remains dependent on the order NK because of the parameters needed by the regime-dependent unconditional covariance matrix. Hopefully, our model can be estimated using step-wise procedure of Engle and Sheppard (2001). Assuming Gaussian innovations, the first stage involves the estimation of K univariate GARCH and corresponds to the volatility component of the log-likelihood :

$$\ell_v(\varphi|\mathbf{r}_t) = -\frac{1}{2}\sum_{t=1}^{T}(K\log(2\pi) + 2\log(|D_t|) + \mathbf{r}'_t D_t^{-2}\mathbf{r}_t) \tag{3.24}$$

These variance estimates of r_t are used to construct the matrix D_t :

$$D_t = \text{diag}\{h_{i,t}^{1/2}\} \tag{3.25}$$

and enable the computation of the standardized residuals :

$$\varepsilon_t = D_t^{-1}\mathbf{r}_t \tag{3.26}$$

In a second step, the correlation component is estimated by maximizing the likelihood conditional upon the volatility parameters estimated in the first step :

$$\ell_c(\theta|\mathbf{r}_t, \varphi) = -\frac{1}{2}\sum_{t=1}^{T}(\log(|R_t|) + \varepsilon'_t R_t^{-1}\varepsilon_t) \tag{3.27}$$

Since the Markov chains s^i for $i = 1,...,M$ are not observed, we shall need to infer the state of these Markov chains by means of a filtration step. Assuming each chain can have N distinct values, our model has M transition matrices of size $N \times N$. As pointed out by Ghahramani and Jordan (1997), any factorial hidden Markov representation can be expressed by a regular hidden Markov model with a single transition matrix. Indeed, each state is factored into multiple states of parallel chains, the regular Markov-Switching representation for the transition is given by the Kronecker product of the transition matrices of the parallel chains, which give a transition matrix of size $N^M \times N^M$ (see appendix B.1 for details). Let's call Υ the resulting transition matrix of the regular Markov-Switching formulation, then in the case of equation (3.19) we have :

$$\Upsilon = \bigotimes_{i=1}^{M} \mathrm{P}^{\mathrm{i}} \tag{3.28}$$

where $\otimes$ denotes the Kronecker product. Estimation can now be performed using standard tools. Since the elements of the transition matrix Υ are a combination of the elements of matrix P^{i}, $i = 1,...,M$, we will denote them by υ_{ij}, with $i, j = 1,...,N^M$. Thus, to perform Hamilton's filter (see Hamilton (1989, 1990, 1994)), we first need to define the probability of an element being in regime j given the information set :

$$\xi_{jt} = \mathbb{P}[\upsilon_t = j | \mathscr{F}_{t-1}, \theta] \tag{3.29}$$

and the density under the regime j :

$$\eta_{jt} = f(\varepsilon_t | \upsilon_t = j, \mathscr{F}_{t-1}; \theta) \tag{3.30}$$

Inference on the state of the Markov chain is then defined by :

$$\hat{\xi}_{t|t} = \frac{(\hat{\xi}_{t|t-1} \odot \eta_t)}{\imath'(\hat{\xi}_{t|t-1} \odot \eta_t)} \tag{3.31}$$

where $\odot$ denotes element-by-element multiplication. $\hat{\xi}_{t|t}$ contains the probability to be in each regime at time t given the observations set up to t, which can be expressed as :

$$\hat{\xi}_{t|t+1} = \Upsilon \times \hat{\xi}_{t|t} \tag{3.32}$$

The regular Markov-Switching representation of a factorial hidden Markov model makes inference on states easier using Hamilton's filter. However, estimation of multivariate GARCH models remains difficult. Moreover, as pointed out by Engle (2009), definition and estimation of the intercept C_{s_t} is the most difficult part of our model. In the standard DCC, Engle and Sheppard (2001) use *variance targeting*. This means that the intercept is estimated via an auxiliary estimator based on a moment condition. In the case of a Markov-Switching representation of DCC, this solution is no longer possible. Hence, the specification of Haas and Mittnik (2008) needs to estimate one intercept matrix per regime. In our case, the the situation is more complicated since we have N^M intercepts to estimate [3].

As point out by Palandri (2009), maximization of the likelihood can be view of a challenge for optimizer, and this is especially true in regime switching MARCH. In this paper, we used Gradient methods based on numerical derivatives. This motivated to take a very parsimonious specification to avoid problems of local maxima.

3. We explain in detail how to specify the intercepts in order to be parsimonious in parameters.

3.2.5 Smoothing

Smoothed probabilities can be compute from the regular Markov-Switching representation. These filtered probabilities refer to inferences on the state conditional on all the information set of the sample. Computation is done using the filter of Kim (1994) given by :

$$\hat{\xi}_{t|T} = \hat{\xi}_{t|t} \odot \{\Upsilon'[\hat{\xi}_{t+1|T} \oslash \hat{\xi}_{t+1|t}]\} \tag{3.33}$$

where $\oslash$ denotes element-by-element division. Nevertheless, filtered probabilities computed with equation (3.33) refer to the regular Markov-Switching representation, i.e. inference of mixed states. In appendix B.2, we propose an easy method to recover smoothed probabilities from those of the regular representation.

3.2.6 Multi-step-ahead correlation matrix forecast

As in the DCC, forecasting the covariance matrix of our model can be decomposed into two separate parts : the forecast of the diagonal matrix D_t on one hand and the forecast of the correlation matrix on the other. The matrices containing the time-varying standard variations can be calculated recursively. Using a GARCH$(1,1)$ model, the volatility forecast at time $T-1$ can be computed such that :

$$h_{t+r} = \sum_{i=1}^{r-2} \omega(\alpha+\beta)^i + (\alpha+\beta)^{r-1} h_{t+1} \tag{3.34}$$

where $h_{t+1} = \omega + \alpha r_t^2 + \beta h_t$ is known at the period t. The second part is far from obvious. The first problem arises with the factorial structure of the chains. An easy way to compute the r-step-ahead forecast of the correlation matrix is to adopt the regular MS representation. It gives :

$$\hat{R}_{T,T+r} = \sum_{\tau=1}^{r} \hat{R}_{T,T+\tau} = \sum_{\tau=1}^{r} \sum_{i=1}^{M^N} \mathbb{P}[\upsilon_\tau = i|\mathscr{F}_{T-1}]\hat{R}^i_{T,T+\tau} \tag{3.35}$$

with $\hat{R}_{T,T+r}$ the correlation matrix forecast from T for the next r steps and :

$$\mathbb{P}[\upsilon_{t+\tau} = i|\mathscr{F}_{T-1}] = \Upsilon^\tau \times \mathbb{P}[\upsilon_t = i|\mathscr{F}_{T-1}] \tag{3.36}$$

The second problem occurs for the formula of covariances forecast. Indeed, the forecasting of the correlation matrix is a combination of the forecasting covariance of the residuals and the square root of residuals. According to Engle and Sheppard (2001), one way to deal this problem is to make an approximation. Assuming that $\overline{V}(s_t^{1,\dots,M}) \approx \overline{R}(s_t^{1,\dots,M})$ and $\mathbb{E}_t[Q_{t+1}(s^{1,\dots,M})] \approx \mathbb{E}_t[R_{t+1}(s^{1,\dots,M})]$, the r-step-ahead correlation matrix forecast is given by :

$$\mathbb{E}_t[R_{t+r}] = \sum_{i=1}^{r-2} \big(1 - a(s^{1,\dots,M}) - b(s^{1,\dots,M})\big)\overline{R}(s^{1,\dots,M})\big(a(s^1) + b(s^{1,\dots,M})\big)^i + \big(a(s^{1,\dots,M}) + b(s^{1,\dots,M})\big)^{r-1} R_{t+1}(s^{1,\dots,M}) \tag{3.37}$$

with $\overline{R}(s^{1,\dots,M})$ being the average correlations of the regimes depending on $s^{1,\dots,M}$. Hence, in each regime, the forecast of the conditional correlation matrix will converge to the long run unconditional correlation matrix of the regime.

3.3 Applications

We apply our model to a dataset of daily data. This dataset contains three exchange rates (Canadian dollar, Yen and Pound) against the US dollar from March 1999 to July 2009. It contains 2697 daily observations (see figure 3.3). We shall consider a comparison of three models : standard DCC with diagonal-BEKK dynamic, Markov-Switching DCC following the Haas and Mittnik (2008) specification with also a diagonal-BEKK dynamic and our proposed model. All time series were downloaded from DataStream and the results are generated using Matlab on Linux.

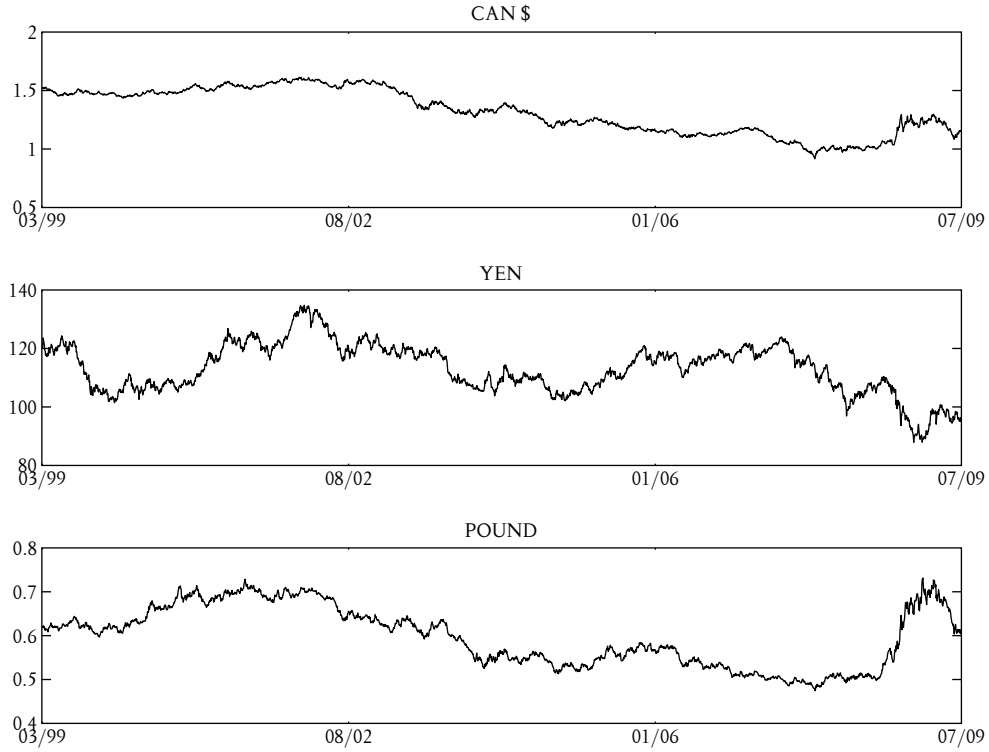


FIGURE 3.3 – Canadian dollar, Yen and Pound from March 1999 to July 2009.

In this application, returns are calculated by taking 100 times the first difference of the logarithm of each series minus the sample mean. Descriptive statistics of log-returns are given in table 3.0(a). These series are then filtered with a GARCH$(1,1)$ model to obtain the standardized residuals. Results are shown in table 3.0(b).

The first model we apply is the DCC with diagonal BEKK dynamic and is written as :

$$Q_t = \overline{P} - \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \overline{P} \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} - \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} \overline{P} \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} + \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \varepsilon_{t-1}\varepsilon_{t-1}' \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}' + \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} Q_{t-1} \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}' \tag{3.38}$$

where $\overline{P}$ refers to the unconditional covariance matrix of the standardized residuals. Using va-

(a) DESCRIPTIVE STATISTICS FOR THE CANADIAN DOLLAR, YEN AND POUND.

	Can $	Yen	Pound
Minimum	-5.7236	-4.9343	-4.4737
Maximum	3.2253	3.2156	3.9187
Mean	-0.0105	-0.0081	-0.0009
Variance	-0.0105	-0.0081	-0.0009
Skewness	-0.3784	-0.4181	0.0656
Kurtosis	10.6898	6.3327	8.4107
Jarque-Bera	$6.7095e+03$ $(1.0000e-03)$	$1.3267e+03$ $(1.0000e-03)$	$3.2918e+03$ $(1.0000e-03)$
Box-Ljung Q(25)	56.15523 $(0.3465e-03)$	29.2504 (0.2536)	80.8896 $(0.8272e-07)$
Box-Ljung Q(50)	103.2819 $(0.0142e-03)$	65.6647 (0.0678)	131.0487 $(0.0344e-07)$

(b) GARCH(1,1) ESTIMATES.

	Can $		Yen		Pound	
Parameters	Estimate	Std. error	Estimate	Std. error	Estimate	Std. error
ω	0.0009	1.821e-07	0.0050	2.84e-06	0.0027	8.53e-07
α	0.0420	3.09e-05	0.0291	2.84e-06	0.0442	4.26e-05
β	0.9559	3.55e-05	0.9596	7.15e-05	0.9477	6.12e-05

(c) DIAG-DCC ESTIMATES.

Parameters	Estimate	Std. error
a_{11}	0.0984	3.2e-4
a_{22}	0.1462	0.0014
a_{33}	0.11030	3.0e-4
b_{11}	0.9938	9.0e-6
b_{22}	0.9869	4.6e-5
b_{33}	0.9937	7.4e-6

(d) MS-DCC.

Parameters	Estimate	standard errors
a_{11}	0.0998	0.0267
a_{22}	0.0712	0.0022
a_{33}	0.0730	0.001
b_{11}	0.9947	4.31e-06
b_{22}	0.9308	0.007
b_{33}	0.9960	2.15e-06
const	$\begin{bmatrix} 5.81e-09 & -7.70e-05 & 2.23e-04 \\ & 0.0085 & 5.12e-04 \\ & & 3.93e-04 \end{bmatrix}$	$\begin{bmatrix} 3.02e-09 & 4.08e-07 & 4.56e-08 \\ & 1.09e-04 & 2.75e-07 \\ & & 2.70e-08 \end{bmatrix}$
a_{11}	0.0120	2.09e-05
a_{22}	0.0300	9.24e-05
a_{33}	0.0676	2.93e-04
b_{11}	0.9896	3.40e-05
b_{22}	0.9902	1.04e-05
b_{33}	0.9914	4.84e-06
const	$\begin{bmatrix} 2.53e-04 & 1.07e-04 & 2.14e-04 \\ & 4.32e-04 & 3.29e-04 \\ & & 0.001 \end{bmatrix}$	$\begin{bmatrix} 1.11e-09 & 1.14e-09 & 3.96e-09 \\ & 3.03e-08 & 7.34e-09 \\ & & 1.16e-06 \end{bmatrix}$
p_{11}	0.9990	3.54e-06
p_{22}	0.9985	9.53e-07

TABLE 3.1 – Descriptive statistics, GARCH(1,1) estimates, DCC and MS-DCC estimated parameters.

riance targeting, $\overline{P}$ can be estimated as :

$$\overline{P} = T^{-1} \sum_{t=1}^{T} \varepsilon_{t-1} \varepsilon_{t-1}' \tag{3.39}$$

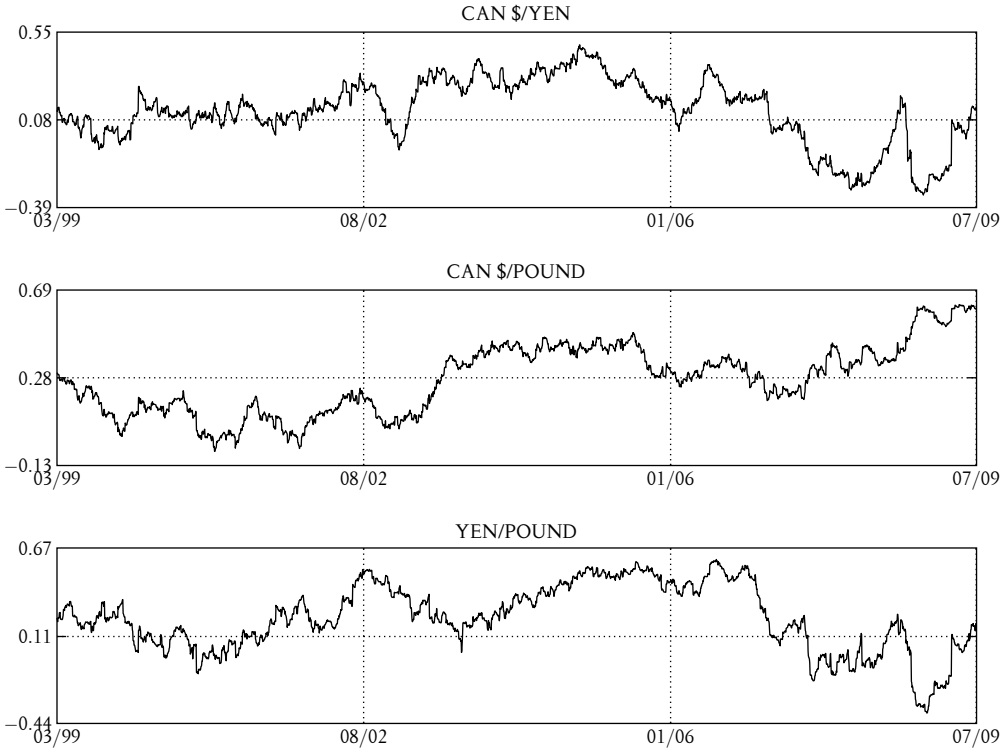


FIGURE 3.4 – Estimated correlations of diagonal DCC.

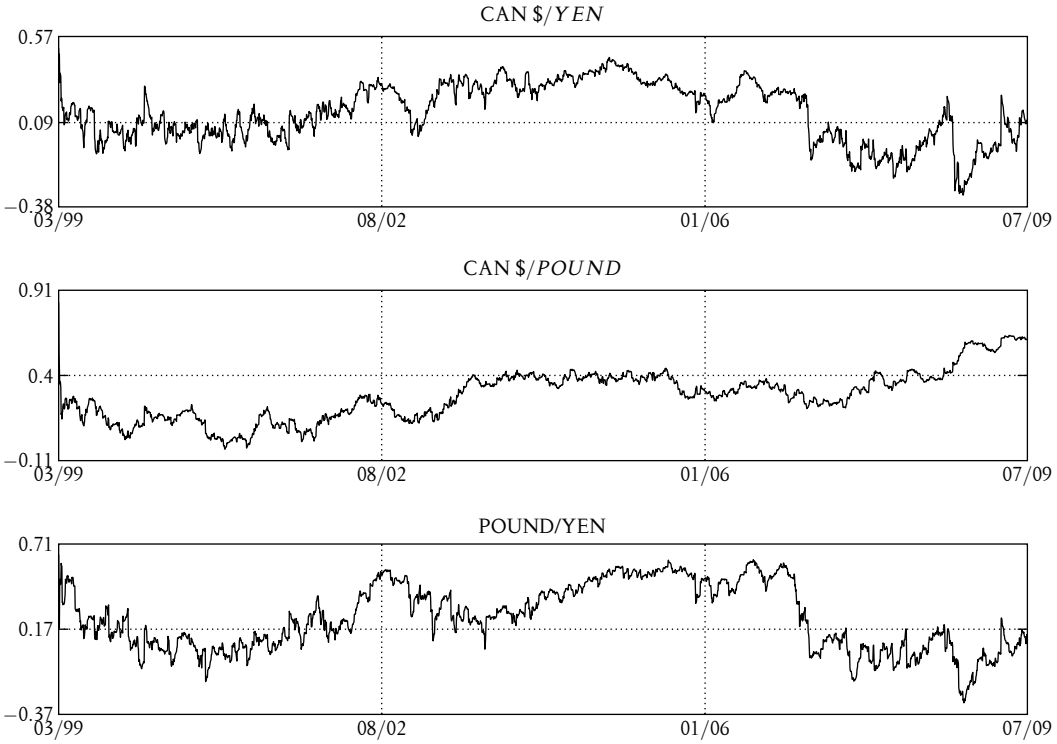
This model is very parsimonious and needs only six parameters. Estimated correlations are plotted in figure 3.4.
The second model we compare is the DCC based on the diagonal-BEKK dynamic of Haas and Mittnik (2008) with two regimes. It can be expressed as :

$$Q_{t,s_t} = \begin{bmatrix} q_{11,s_t} & q_{12,s_t} & q_{12,s_t} \\ & q_{22,s_t} & q_{23,s_t} \\ & & q_{33,s_t} \end{bmatrix} + \begin{bmatrix} a_{11,s_t} & 0 & 0 \\ 0 & a_{22,s_t} & 0 \\ 0 & 0 & a_{33,s_t} \end{bmatrix} \varepsilon_{t-1}\varepsilon'_{t-1} \begin{bmatrix} a_{11,s_t} & 0 & 0 \\ 0 & a_{22,s_t} & 0 \\ 0 & 0 & a_{33,s_t} \end{bmatrix}' + \begin{bmatrix} b_{11,s_t} & 0 & 0 \\ 0 & b_{22,s_t} & 0 \\ 0 & 0 & b_{33,s_t} \end{bmatrix} Q_{t-1,s_t} \begin{bmatrix} b_{11,s_t} & 0 & 0 \\ 0 & b_{22,s_t} & 0 \\ 0 & 0 & b_{33,s_t} \end{bmatrix}' \quad (3.40)$$

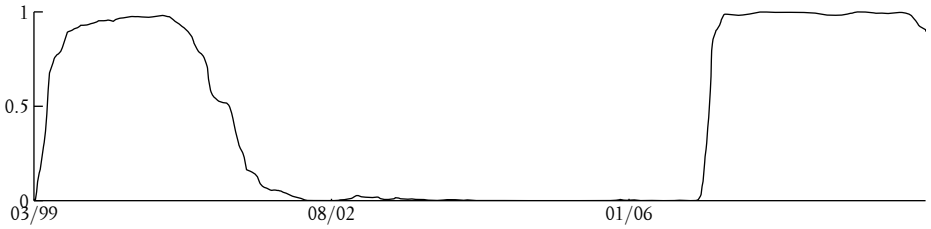
where s_t is an unobserved first-order Markov chain process with a 2-by-2 transition matrix :

$$\mathrm{P} = \begin{bmatrix} p_{11} & 1-p_{22} \\ 1-p_{11} & p_{22} \end{bmatrix} \quad (3.41)$$

with $p_{ij} = \mathbb{P}[s_{t+1} = j | s_t = i],\ i, j = 1, 2$. In that case, variance targeting is not feasible and Q_{t,s_t} needs 7 parameters for the constant, 6 for the coefficients of the dynamic, and 2 more parameters for the transition matrix. For two regimes, this makes a total of 28 parameters. Estimation was done using Gradient methods, filtered probabilities are computed with Hamilton's filter, and smoothed probabilities are calculated with Kim's filter. Results appear in figure 3.5. The model clearly identifies two regimes. The switch from one to the other coincided with the coming of the so-called *subprime crisis*. In comparison with the previous DCC model, this model reveals superior explanatory power.



(a) ESTIMATED CORRELATIONS.



(b) SMOOTHED PROBABILITIES.

FIGURE 3.5 – Estimated correlations and smoothed probabilities of the multivariate Markov-Switching DCC.

We now apply the specification presented in this paper. With three series, our model has the following representation :

$$Q_{t,s_t^{1,2,3}} = \overline{Q}_{s_t^{1,2,3}} + \begin{bmatrix} a_{11,s_t^1} & 0 & 0 \\ 0 & a_{22,s_t^2} & 0 \\ 0 & 0 & a_{33,s_t^3} \end{bmatrix} \varepsilon_{t-1}\varepsilon_{t-1}' \begin{bmatrix} a_{11,s_t^1} & 0 & 0 \\ 0 & a_{22,s_t^2} & 0 \\ 0 & 0 & a_{33,s_t^3} \end{bmatrix}' + \begin{bmatrix} b_{11,s_t^1} & 0 & 0 \\ 0 & b_{22,s_t^2} & 0 \\ 0 & 0 & b_{33,s_t^3} \end{bmatrix} Q_{t-1,s_t^{1,2,3}} \begin{bmatrix} b_{11,s_t^1} & 0 & 0 \\ 0 & b_{22,s_t^2} & 0 \\ 0 & 0 & b_{33,s_t^3} \end{bmatrix}' \tag{3.42}$$

where s_t^1, s_t^2 and s_t^3 are three unobserved first-order Markov chain processes with transition matrix :

$$\mathrm{P^i} = \begin{bmatrix} p_{11}^i & 1-p_{22}^i \\ 1-p_{11}^i & p_{22}^i \end{bmatrix} \tag{3.43}$$

for $i = 1,2,3$. With three chains of two states, the factorial decomposition of the process, the

model allows for eight combinations of the regimes. It implies that we need eight intercept matrices. There are two ways to specify these ones. A naive approach is to estimate the eight full matrices for the intercepts. But this solution is difficult from a numerical point of view. Another way to estimate these matrices is to define the intercept corresponding of the two extreme cases (all covariances are in regime 1, and all are in regime 2), and then construct the other six intercept matrices. Let :

$$\overline{Q}_{s_t^1=1,s_t^2=1,s_t^3=1} = \begin{bmatrix} q_{11,s_t^1=1} & q_{12,s_t^1=1,s_t^2=1} & q_{13,s_t^1=1,s_t^3=1} \\ & q_{22,s_t^2=1} & q_{23,s_t^2=1,s_t^3=1} \\ & & q_{33,s_t^3=1} \end{bmatrix} \tag{3.44}$$

the constant corresponding to the case where all the covariances are in regime 1 and :

$$\overline{Q}_{s_t^1=2,s_t^2=2,s_t^3=2} = \begin{bmatrix} q_{11,s_t^1=2} & q_{12,s_t^1=2,s_t^2=2} & q_{13,s_t^1=2,s_t^3=2} \\ & q_{22,s_t^2=2} & q_{23,s_t^2=2,s_t^3=2} \\ & & q_{33,s_t^3=2} \end{bmatrix} \tag{3.45}$$

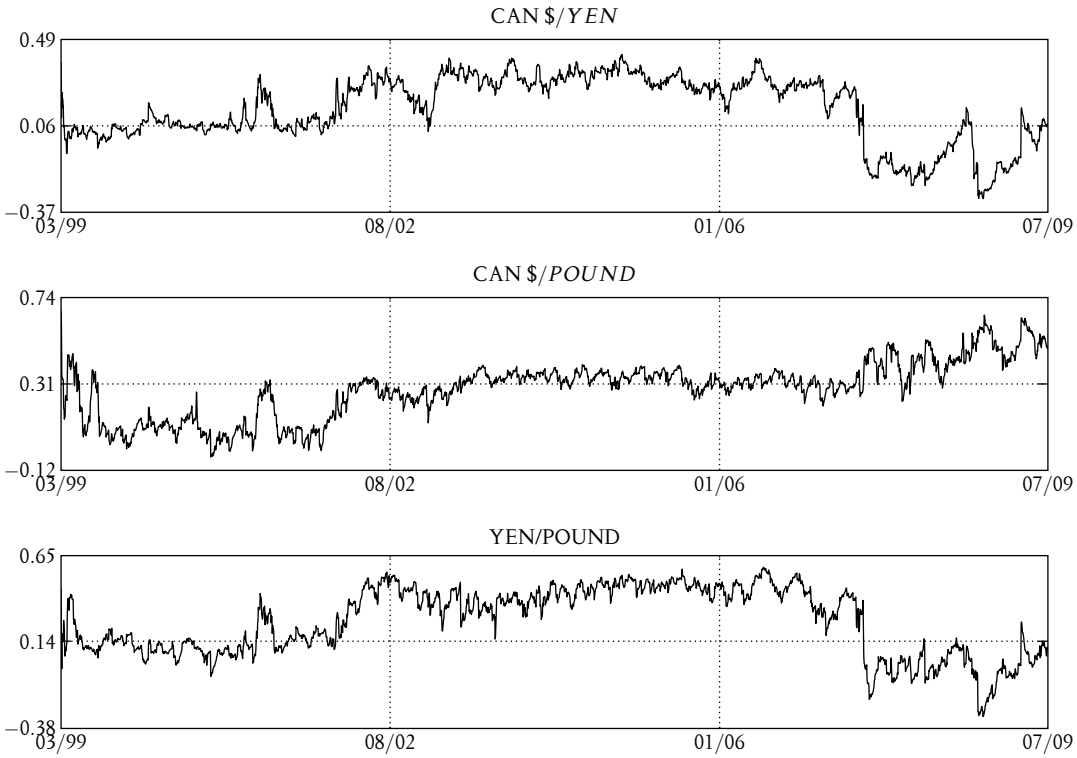
the intercept where all covariances are in regime 2. Given time series are ordered as CAD, YEN and POUND, the constant corresponding to the case where the covariance of CAD/YEN is in regime 1, CAD/POUND in regime 2 and YEN/POUND in regime 1 can be written using the parameters of the previous matrix :

$$\overline{Q}_{s_t^1=1,s_t^2=2,s_t^3=1} = \begin{bmatrix} q_{11,s_t^1=1} & \boxed{q_{12,s_t^1=1,s_t^2=2}} & q_{13,s_t^1=1,s_t^3=1} \\ & q_{22,s_t^2=2} & \boxed{q_{23,s_t^2=2,s_t^3=1}} \\ & & q_{33,s_t^3=1} \end{bmatrix} \tag{3.46}$$

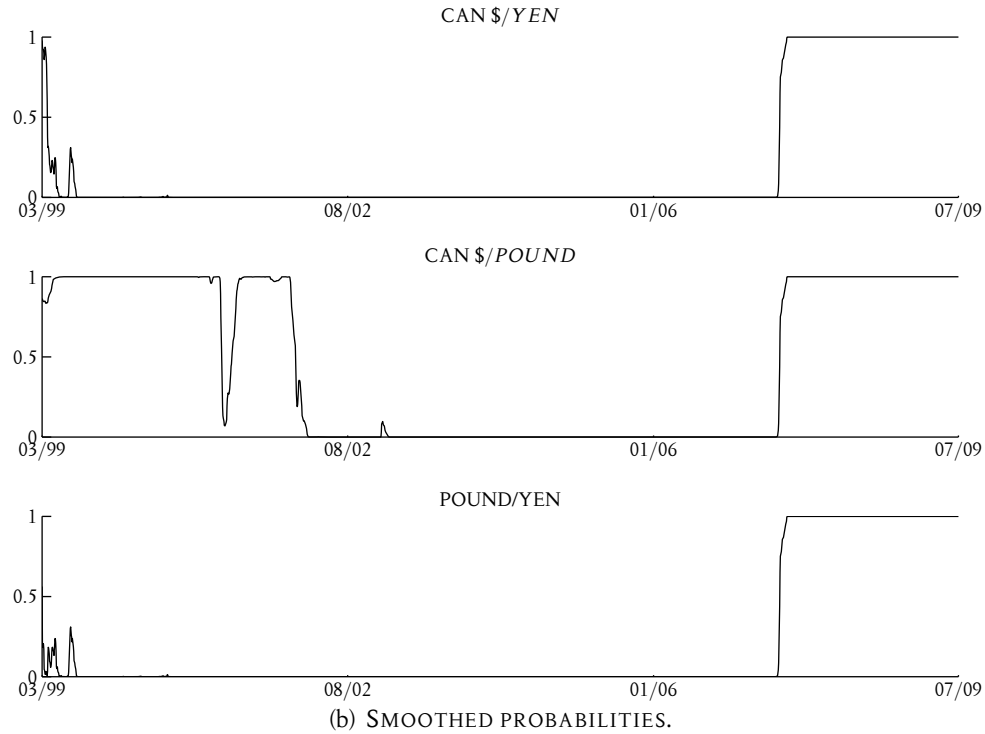
Non-boxed elements are coming from the intercept of the two extreme cases. Boxed elements corresponds to the additional elements needed to define the intercept in that case. The constant corresponding to the case where the covariance CAD/YEN and CAD/POUND are in regime 1 and YEN/POUND in regime 2 can be written using the parameters the previous matrix :

$$\overline{Q}_{s_t^1=1,s_t^2=1,s_t^3=2} = \begin{bmatrix} q_{11,s_t^1=1} & q_{12,s_t^1=2,s_t^2=1} & \boxed{q_{13,s_t^1=2,s_t^3=2}} \\ & q_{22,s_t^2=1} & \boxed{q_{23,s_t^2=2,s_t^3=2}} \\ & & q_{33,s_t^3=2} \end{bmatrix} \tag{3.47}$$

where boxed terms refer to the two additional elements to be estimated. By doing so for the other matrices, we need $14+6\times 2$ parameters (the complete formulation of the intercept matrices are given in appendix B.3). Given the three transition matrices need 3×2 parameters and the dynamic elements 12×2, making the total number of parameters of our model is 42. Estimation is done using the regular HMM representation. This strategy has the advantage that it allows the use of standard (Hamilton filter and Kim's smoothing), but has the disadvantage of being a complex optimization problem. To avoid the problem of the local maxima, we have performed twenty estimations with various starting points. Our computer, with a Q6600 processor, took approximately twelve hours to complete a single estimate. As pointed out by Engle (2009), the most complicated part of the estimation process is the intercept matrices. Because variance targeting is not possible, we have to estimate eight intercept matrices. Even with the strategy explained previously, the numerical constraints needed to keep these eight matrices positive definite make the maximization of the likelihood difficult. Results are plotted in figure 3.6 and estimated parameters are in table 3.2.
A comparison of the smoothed probabilities plotted in figures 3.5(b) and 3.6(b) will reveal the differences between the classical Markov-Switching approach and the factorial decomposition.



(a) ESTIMATED CORRELATIONS.



(b) SMOOTHED PROBABILITIES.

FIGURE 3.6 – Estimated correlations and smoothed probabilities of the multivariate Markov-Switching DCC with FHMM specification.

With one Markov chain for all the conditional covariances, the DCC based on the specification of Haas and Mittnik (2008) identifies two regimes that appear once. A break occurs at the beginning

Parameters	Estimate	standard errors
a^1_{11}	0.1936	0.0124
a^1_{22}	0.1112	0.00556
a^1_{33}	0.3293	0.0080
a^2_{11}	0.0334	0.0092
a^2_{22}	0.2683	0.0163
a^2_{33}	0.1610	6.64e-04
b^1_{11}	0.9760	1.87e-04
b^1_{22}	0.9835	0.0163
b^1_{33}	0.9389	2.41e-05
b^2_{11}	0.9874	0.0049
b^2_{22}	0.9581	2.26e-04
b^2_{33}	0.9819	5.70e-04
$\overline{Q}_{s^1_t=1,s^2_t=1,s^3_t=1}$	$\begin{bmatrix} 0.0395 & -0.0025 & 0.0648 \\ & 0.0320 & 0.0087 \\ & & 0.1544 \end{bmatrix}$	$\begin{bmatrix} 2.84e{-}07 & 4.67e{-}05 & 5.25e{-}05 \\ & 9.10e{-}06 & 9.55e{-}05 \\ & & 2.08e{-}04 \end{bmatrix}$
$\overline{Q}_{s^1_t=1,s^2_t=1,s^3_t=2}$	$\begin{bmatrix} --- & --- & 0.0648 \\ --- & --- & 0.0087 \\ --- & --- & --- \end{bmatrix}$	$\begin{bmatrix} --- & --- & 1.59e{-}06 \\ --- & --- & 5.63e{-}05 \\ & & --- \end{bmatrix}$
$\overline{Q}_{s^1_t=1,s^2_t=2,s^3_t=1}$	$\begin{bmatrix} --- & 0.0549 & --- \\ --- & --- & 0.1248 \\ --- & --- & --- \end{bmatrix}$	$\begin{bmatrix} --- & 9.00e{-}05 & --- \\ --- & --- & 0.0051 \\ --- & --- & --- \end{bmatrix}$
$\overline{Q}_{s^1_t=1,s^2_t=2,s^3_t=2}$	$\begin{bmatrix} --- & -0.0466 & 0.0259 \\ --- & --- & --- \\ --- & --- & --- \end{bmatrix}$	$\begin{bmatrix} --- & 1.57e{-}05 & 5.81e{-}04 \\ --- & --- & --- \\ --- & --- & --- \end{bmatrix}$
$\overline{Q}_{s^1_t=2,s^2_t=1,s^3_t=1}$	$\begin{bmatrix} --- & 0.0527 & -0.0312 \\ --- & --- & --- \\ --- & --- & --- \end{bmatrix}$	$\begin{bmatrix} --- & 0.0076 & 7.68e{-}04 \\ --- & --- & --- \\ --- & --- & --- \end{bmatrix}$
$\overline{Q}_{s^1_t=2,s^2_t=1,s^3_t=21}$	$\begin{bmatrix} --- & -0.0278 & --- \\ --- & --- & 0.0144 \\ --- & --- & --- \end{bmatrix}$	$\begin{bmatrix} --- & 4.08e{-}07 & --- \\ --- & --- & 2.75e{-}07 \\ --- & --- & --- \end{bmatrix}$
$\overline{Q}_{s^2_t=1,s^2_t=2,s^3_t=1}$	$\begin{bmatrix} --- & --- & 0.0636 \\ --- & --- & 0.1200 \\ --- & --- & --- \end{bmatrix}$	$\begin{bmatrix} --- & --- & 0.0968 \\ --- & --- & 0.0046 \\ --- & --- & --- \end{bmatrix}$
$\overline{Q}_{s^1_t=2,s^2_t=2,s^3_t=2}$	$\begin{bmatrix} 0.1551 & 0.0885 & -0.0085 \\ & 0.1205 & 0.0033 \\ & & 0.0859 \end{bmatrix}$	$\begin{bmatrix} 6.20e{-}04 & 0.0012 & 0.0011 \\ & 4.69e{-}04 & 0.0052 \\ & & 0.0018 \end{bmatrix}$
p^1_{11}	0.9830	5.70e-04
p^1_{22}	0.4016	0.0375
p^2_{11}	0.9867	9.02e-04
p^2_{22}	0.9860	5.41e-04
p^3_{11}	0.8179	0.0096
p^3_{22}	0.4729	0.0425

TABLE 3.2 – FHMM-DCC estimated parameters.

of the subprime crisis. On the other hand, our specification, with one Markov chain for each conditional covariance, clearly shows it might be unrealistic to assume that all covariances have the same switching dynamic. If all the conditional covariances switch at the beginning of the subprime crisis, it is worth noting they have truly distinct dynamics. This comparison clearly establishes that relaxing the constraint of common switching dynamic provides additional explanatory power.

3.4 Conclusion

In this paper, we have presented a generalization of the DCC model. Our model is based on the Markov-Switching method introduced by Haas and Mittnik (2008). We have suggested a specification based on the factorial hidden Markov model (FHMM) of Ghahramani and Jordan (1997). Thus, in our model, each covariance of the conditional covariance matrix has its own dynamic. Consequently, our switching dynamics relax the classical assumption of Markov-Switching models where all the conditional covariances switch together from one regime to another. While the results of the application in this paper may contribute to a better understanding of the dynamic of the correlations of exchange rate data, we must point out that several limitations must yet to be overcome. The most important of these is certainly the estimation of the model. The large number of time series that our specification calls for raises a complex numerical problem. These tasks await future research.

CHAPITRE

4

Modelling Volatility and Correlations with a Hidden Markov Decision Tree.

4.1 Introduction

Multivariate volatility models have been a field of very active research during the past two decades. Related to this topic, the study of multivariate GARCH with conditional correlations has attracted a strong interest since the seminal paper of Bollerslev (1990). The introduction of dynamics in the conditional correlations has led this pioneering approach to the creation of a new class of models, the well-known Dynamic Conditional Correlation (DCC) models. Proposed by Engle and Sheppard (2001) and in a slightly different manner by Tse and Tsui (2002), the DCC provides an attractive means of parametrization.

This new class now includes many extensions. Engle (2002) generalized the model by introducing full matrices instead of scalars to drive the dynamics of the process (Generalized DCC). Therefore, this unrestricted specification requires a large number of parameters, and makes estimation difficult. Various extensions have been proposed in order to overcome the problems of dimensionality that accompany attempts to relax constraints on the dynamics. Most of them are based on combinations of specific matrix, partitioned vectors or block matrices (see Billio and Caporin (2009), Hafner and Franses (2009) and Billio, Caporin, and Gobbo (2006)), and clustering techniques (see Zhou and Chan (2008)). A DCC incorporating exogenous variables has been proposed by Vargas (2008) with which to identify factors that could lead to correlations. Franses, Hafner, and van Dijk (2005) present a semi-parametric approach using a transformed Nadaraya-Watson estimator. Feng (2007) also presents a local estimator for the correlations based on k-nearest-neighbors (k-NN) methods.

Others extensions focus on asymmetric effects in the conditional correlations. An asymmetric extension has been proposed by Cappiello, Engle, and Sheppard (2006) to help in the study of worldwide linkages in the dynamics of the correlations of selected bonds and equity markets. Asymmetry here refers to the upward motion of volatility more pronounced after a negative shock than after a positive shock. Vargas (2006) extends the Block DCC of Billio, Caporin, and Gobbo (2003) to take account of the asymmetry between blocks of asset returns. Cajigas and Urga (2006) also propose an asymmetric generalized DCC where standardized residuals follow an asymmetric multivariate Laplace distribution.

Another direction such extensions have taken is to be seen in the regime switches introduced in the conditional correlations in order to facilitate handling empirical findings in periods of turbulence. In this context, Silvennoinen and Teräsvirta (2005) assume that the conditional correlation matrix varies smoothly between two constant correlations matrices. The link between these two extreme matrix extreme types of matrix is established via a conditional logistic function using an exogenous or endogenous transition variable. This model has been further improved to allow another transition around the first one (see Silvennoinen and Teräsvirta (2009)). In a departure from his previous STAR approach, Pelletier (2006) sought to introduce Markov switches in the correlations. In this case, the correlations are constant within each regime, but may vary from one regime to another. Both these approaches, however, assume that the correlations evolve between constant correlation matrices. Billio and Caporin (2005) explain a strategy in which the scalar DCC of Engle and Sheppard (2001) is extended to the case in which both the parameters and the unconditional correlation are driven by a hidden Markov chain.

Alongside these developments, the technical issue of the curse of dimensionality, which underlies most of the developments in the field of multivariate GARCH, has been the subject of recent advances. This problem of the explosion in the number of parameters can become a daunting challenge because of its size, and its tendency to grow exponentially : with k time series the number of correlations to estimate is $k(k-1)/2$. Several strategies have been proposed to resolve the issue. Palandri (2009) decomposes the correlations and partial correlations. This decomposition transforms a high dimensional optimization problem into a set of simple estimates. Engle, Shephard,

and Sheppard (2008) also exploited the idea of decomposing a complex estimation problem into a set of simpler models to construct a composite likelihood which is obtained from the summation of the likelihoods of subsets of assets. Engle and Kelly (2009) suggest a radically different strategy based on an updated approximation of each correlation. In each instance, they assume that the paired correlations are all equal. Engle (2008) had earlier advanced an alternative approach also based on approximations. In the so-called *MacGyver method*, based on the bivariate estimation of each pair of correlations, one simply calculates the median of these estimators.

This brief summary of the literature is far from exhaustive, and we would refer the interested reader to Bauwens, Laurent, and Rombouts (2006) for a general survey of multivariate GARCH. The paper of Silvennoinen and Teräsvirta (2009) contains a useful review of recent advances, and Engle (2009) has an exhaustive survey of conditional correlation models.

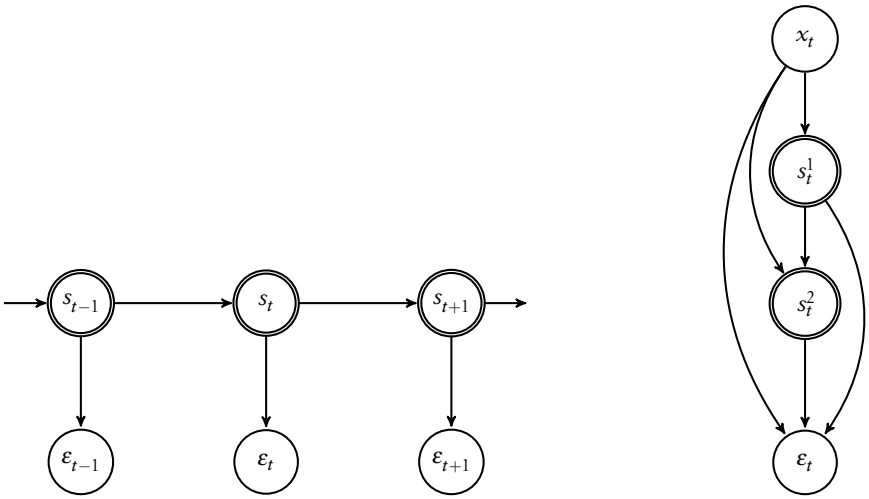
The model we present is an extension of multivariate GARCH with conditional correlations based on a tree structure. Models based on tree structure have already been proposed by other writers. Barone-Adesi and Audrino (2006) utilized the tree structure concept developed by Audrino and Buhlmann (2001) and Audrino and Trojani (2006) to arrive at a rolling window averaged conditional correlation estimator. This method allows thresholds in conditional correlations. The estimator is a convex combination of realized correlations and estimates of averaged correlations. As the averaged conditional correlation is constructed with univariate GARCH models, this approach has that it only requires a small number of parameters. It has been extended using non-parametric functions based on functional gradient descent to estimate univariate volatilities by Audrino (2006). Also based on the idea of Audrino and Buhlmann (2001), the model suggested by Trojani and Audrino (2005) adopts the two-step procedure explained in Engle and Sheppard (2001). In first time, conditional variances are extracted with tree-structured GARCH models. Conditional correlations are then computed from standardized residuals. These correlations have their own tree-structured dynamic. An tree-structured extension of the DCC is suggested by Dellaportas and Vrontos (2007) suggest a tree-structured extension of the DCC as way to study volatility and co-volatility asymmetries. Making a link between conditional variances and correlations, this model permits the appearance of multivariate thresholds. Using a sequence of binary decision rules, each terminal node is matched with a multivariate GARCH model.

A common feature of these approaches is that they are all based on the binary tree principle. Time series are recursively partitioned using binary decisions. The resulting tree is purely deterministic. In contrast, we propose in this paper an extension of the DCC based on a stochastic decision tree. Our model links the univariate volatilities with the correlations via a hidden stochastic decision tree. The ensuing Hidden Markov Decision Tree (HMDT) model is in fact an extension of the Hidden Markov Model (HMM) introduced by Jordan, Ghahramani, and Saul (1997). Based on a Markov temporal structure, the architecture of this model is the opposite of the classical deterministic approach based on a binary decision tree. Our model can be estimated via maximum likelihood methods. The paper is organized as follows. The section 4.2.1 recalls the basics of hidden Markov decision tree. Our model is defined in section 4.2.2. In section 4.3, the model is applied to a bivariate dataset. Section 4.4 carries the concluding remarks, and proposes areas for further research.

4.2 The Model

4.2.1 What is a Hidden Markov Decision Tree ?

The approach that we have adopted in this paper has been developed around the idea of a hierarchical architecture with a Markov temporal structure. This model can be explained in one of two ways.



(a) HIDDEN MARKOV MODEL.

(b) HIERARCHICAL MIXTURE OF EXPERTS.

FIGURE 4.1 – The Hidden Markov Model (fig.) and the Hierarchical Mixture-of-Experts Model (fig.) as graphical models.

A Hierarchical Mixtures-of-Experts Model with Markov dependencies

The approach can be seen in the first instance as a hybrid model that blends the Hierarchical Mixture-of-Experts (HME) of Jordan and Jacobs (1994) and the Hidden Markov Models (HMM). The HME is based on the *divide-and-conquer* algorithm. This methodology is to fit the data by dividing the space into nested subspaces and estimate a sub-model for each of these regions (see figure 4.1 for the representation of HME as a directed acyclic graph). The data are fitted by a model which is the overall combination of the constituent sub-models. This philosophy has given rise to numerous approaches. Some of which are, without being exhaustive, the CART (see Breiman, Friedman, Olshen, and Stone (1984)), MARS (see Friedman (1991)) and GUIDE (see Loh (2002)). Itself a generalization of the Mixture-of-Experts (ME) model of Jacobs, Jordan, Nowlan, and Hinton (1991), the HME assumes that the data can be fitted by a piecewise function. The approach is based on two elements. First, the model has n *experts* to fit the data in n regions. An expert can be a constant, AR or GARCH process. The second element is a *gating network* that pairs each region with an expert, given an input variable. The final output is is a convex combination of the expert outputs given the input. The ME architecture has been applied to time series by Carvalho and Tanner (2006). While the ME architecture only possesses a single level of experts associated with each gating network, the HME generalizes this approach by allowing different levels of experts to be associated with various gating functions. The HME generalize this approach by allowing different level of experts matched by various gating functions. Huerta, Jiang, and Tanner (2003) used a two-layer HME in their study of the US industrial production index with trend- stationary process experts and difference-stationary process experts. As explained by McLachlan and Peel (2000), the CART/MARS/GUIDE approaches and the HME approach differ principally in that the former use hard boundaries while the latter divides the covariate space with probabilistic schemes. One might also point out the link between the ME architecture and the (S)TAR models. Indeed, in the case of a STAR GARCH model, the transition function can be interpreted as the gating network, and the two GARCH components as the experts. A similar comparison is true for the HME and the multiple-regime STAR of van Dijk and Franses (1999).

The Hidden Markov Model has emerged as a standard tool in modelling time series with regime

switching. This approach has been extensively used, both in graphical models (see Rabiner (1989)) and in econometrics with its counterpart Markov-switching (see among others Hamilton (1994)). The approach is based on the assumption that each observation is linked to a finite number of hidden states via a probability distribution (see figure 4.1). The conditional probability distribution of the hidden variable is generally assumed to be a first-order Markov chain. In time series analysis, this was first used by Hamilton (1989) in connection with with a regime-switching AR process, and later generalized by Krolzig (1997) to the VAR case. Examples of the application to GARCH models can be found in Hamilton and Susmel (1994), Gray (1996), Dueker (1997), Klaassen (2002) and Haas, Mittnik, and Paollela (2004b). Markov-switching-based extensions of multivariate GARCH models with dynamic correlations have also been proposed. Billio and Caporin (2005) proposed a DCC model with Markov switch in the unconditional correlation process. Application of this MS-DCC model to daily stock market indices from January 2000 to December 2003 highlights the advantage of introducing Markov jumps during periods of financial turbulence. Pelletier (2006) also suggested a Markov-Switching DCC. This Regime Switching for Dynamic Correlations (RSDC) model assumes that correlations are constant within each regime, but vary from regime to regime. The HMDT can be seen as a hybrid model combining the two previous approaches. The result gives a probabilistic decision tree in which the decision each time is conditional upon the decision at the previous time.

Factorial and Coupled HMM

We shall now view the HMDT from another standpoint, treating this architecture as an extension of a pure HMM, unlike the hierarchical mixture with Markovian dynamics, the basis of the earlier model. Indeed, the HMDT can be regarded both as a factorial HMM and as a coupled HMM.
The factorial hidden Markov model (FHMM) was proposed by Ghahramani and Jordan (1997). An extension of the basic HMM, this model assumes that each state variable is factored into several state variables, each of them with its own independent Markovian dynamic, and that the output is the combination of the several processes that underlie the state variables (see figure 4.2(a)). Because the model has various independent hidden Markov models in parallel, the resulting state space of the model is the Cartesian product of the parallel sub-processes. The FHMM has been widely used in speech recognition because of its ability to model time series generated from various independent sources (see among others Roweis (2000) for an application to microphone source separation). The distributed state representation of the FHMM also provides an interesting framework for handwriting recognition (see Williams, Toussaint, and Storkey (2007) for the use of a primitive model based on FHMM to represent handwriting data).
A coupled hidden Markov model (CHMM) is an extension of the classical HMM. It is typically a model with several HMMs whose Markov chains interact together (see Brand (1997)). There are two ways of allowing the states to have interdependencies. One way is to designate as *independent coupling* that condition in which various processes coupled at the output. In this case, the FHMM can be considered an independent coupled HMM. In the other case, which is termed *dependent coupling*, the processes are assumed to be dependent, and the current state of each state is dependent on the other states of its own chain as well as on the previous state of another chain, and so on (see figure 4.2(b)). Dependent coupled hidden Markov models include different kinds of architecture, such as models with differing lags, nested loops, and non-linear dependency relationships. Coupled HMMs have been used in various fields like facial event mining (see Ma, Zhou, Celenk, and Chelberg (2004)), video-realistic speech animation (see Xie and Liu (2007)), audiovisual speech recognition (see Nefian, Liang, Pi, Xiaoxiang, Mao, and Murphy (2002)).
In this framework, an HMDT can be view as an HMM which is both factorial and dependent coupled. The factorial decomposition provides a factorized state space. This state space decomposition is done using constant or state dependent time-varying transition probabilities given an

input variable. As described in Brand (1997), the top level of the tree can be seen as a *master* process and the following levels as *slave* processes. The constraint of a level on the following is done via a coupling transition matrix which produces the ordered hierarchy of the structure. As the links between decision states are driven with Markovian dynamics, and the switch from one level to the following is done via a coupling transition matrix, this architecture gives a fully probabilistic decision tree.

4.2.2 Hidden Markov decision tree for correlations

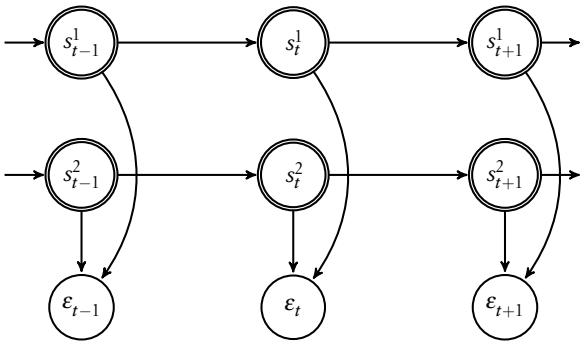
Starting point

The model we propose in this paper is an extension of the DCC of the scalar DCC (see Engle and Sheppard (2001)). Given y_t an K dimensional time series of length T, the DCC class model assumes the form :

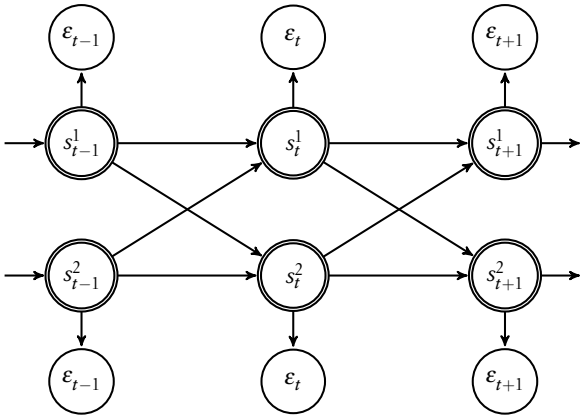
$$y_t|\mathscr{F}_{t-1} \overset{iid}{\sim} \mathscr{N}(0, H_t) \tag{4.1}$$

where $\mathscr{F}_{t-1}$ refers to the information set at time $t-1$. The conditional variance-covariance matrix of returns y_t is expressed as follows :

$$H_t = D_t R_t D_t \tag{4.2}$$



(a) FACTORIAL HIDDEN MARKOV MODEL.



(b) COUPLED HIDDEN MARKOV MODEL.

FIGURE 4.2 – Factorial (4.2(a)) and Coupled HMM (4.2(b)) as graphical models.

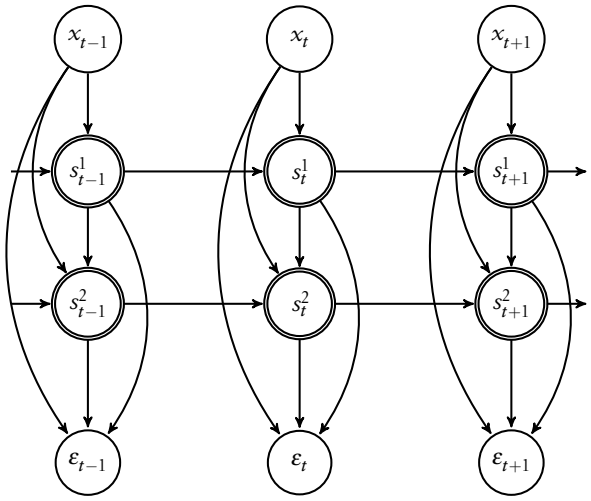


FIGURE 4.3 – The hidden Markov decision tree model viewed as a directed acyclic graph.

where R_t is a $K \times K$ constant correlation matrix. the matrix D_t is a $K \times K$ diagonal matrix containing univariate time varying standard deviations :

$$D_t = \text{diag}\{h_{i,t}^{1/2}\} \tag{4.3}$$

for $i = 1, ..., K$. Getting the matrix D_t is generally referred to the so-called *degarching* filtration required to construct standardized residuals expressed as :

$$\varepsilon_t = D_t^{-1} r_t \tag{4.4}$$

The conditional correlations are simply the expectation of the standardized residuals :

$$\mathbb{E}_{t-1}[\varepsilon_t \varepsilon_t'] = D_t^{-1} H_t D_t^{-1} = R_t \tag{4.5}$$

In the scalar DCC of Engle and Sheppard (2001), correlations are computed using a combination of the conditional covariance of the standardized residuals $Q - t$ which follows a scalar BEKK process :

$$Q_t = (1 - a - b)\overline{V} + a\varepsilon_{t-1}\varepsilon_{t-1}' + bQ_{t-1} \tag{4.6}$$

where $\overline{Q}$ is the unconditional covariance matrix of the standardized residuals and a and b are two positive scalars such that $a + b < 1$. The conditional correlation matrix is then obtained by :

$$R_t = \text{diag}\{Q_t\}^{-1/2} Q_t \text{diag}\{Q_t\}^{-1/2} \tag{4.7}$$

Our model has the DCC of Engle and Sheppard (2001) as its starting point. We propose to improve this specification by introducing a hidden decision. The two levels of the tree help in a study of the relationship between univariate volatility and correlation. The first level sets up a rule to classify the variances as low-regime or high-regime, while the second level sets up a rule to distinguish high-correlation regimes from the low-correlation regimes. Clearly, this construction offers a useful way to investigate the link between high/low volatility and high/low correlations. Specifically, the method allows a study of the probability that a correlation falls in a particular regime category given the regime of the univariate conditional variance.

Hidden tree structure

As pointed out in Section 4.2.1, an HMDT can be viewed either as a factorial HMM or as a coupled HMM. Decomposition of the factorial HMM is used to divide the space of the time series into low and high conditional variances for each series, and low and high for the sequences of correlations. The transition matrix used in our model can be either static or time-varying, and can depend on either endogenous or exogenous variables. Formally, each time series has its own conditional variance driven by a transition matrix with time-varying probabilities. Univariate high and low conditional variances follow a two-state first-order Markov process.
Formally, at the first level, each time series have its own conditional variance driven by a transition matrix with time varying probabilities. Univariate high and low conditional variance follow a two state Markov process of first order such that :

$$\mathbb{P}[s_{t+1} = i | s_t = j, ..., s_t = h] = \mathbb{P}[s_{t+1} = i | s_t = j] \tag{4.8}$$

To introduce time-varying transition probabilities, we can use the specification proposed by Diebold, Lee, and Weinbach (1994). Then, transition probabilities are assumed to follow a logistic function of an endogenous or exogenous variable. Elements of the transition matrix $\mathrm{P}^{\mathrm{k}}_{\mathrm{vol}}$ for the conditional variance of the k^{th} with row i and column j is defined as :

$$p^k_{ij}(t) = \mathbb{P}[s_{t+1} = j | s_t = i; x_{t-1}, \gamma^k_n], \quad i, j = 1, .., N \tag{4.9}$$

Introducing time-variation, transition probabilities are then expressed as :

$$p^k_{ij}(t) = \frac{\exp(x'_{t-1}\gamma^k_n)}{1 + \exp(x'_{t-1}\gamma^k_n)} \tag{4.10}$$

where x_{t-1} is the conditioning vector of input in the figure 4.3 and γ its coefficient. As we partition the space of the univariate conditional variance in two subspaces, low and high variance, the transition matrix for the variance of the k^{th} time series is of size 2×2 and can be expressed as follows :

$$\mathrm{P}^{\mathrm{k}}_{\mathrm{vol}} = \begin{bmatrix} p^k_{11}(t) & 1 - p^k_{22}(t) \\ 1 - p^k_{11}(t) & p^k_{22}(t) \end{bmatrix} \tag{4.11}$$

Since all univariate volatility processes have the same Markovian dynamic specification, individual transition matrices can be aggregated to constitute the first level of the decision tree. Individual HMMs are aggregated in a factorial HMM representation. The dynamic of the univariate volatility level can be resumed in a general transition $\mathrm{P}_{\mathrm{vol}}$ by the cross product of the transition matrices of univariate volatility models $\mathrm{P}^{\mathrm{k}}_{\mathrm{vol}}$:

$$\mathrm{P}_{\mathrm{vol}} = \bigotimes_{i=1}^{K} \mathrm{P}^{\mathrm{i}}_{\mathrm{vol}} \tag{4.12}$$

This factorial representation allows a representation of all the dynamics of the K univariate volatilities containing 2 states with a single transition matrix of size $2^K \times 2^K$.
The same specification is used for the second level. This level discriminates between low and high correlations. Thus, the decision step is represented by a 2-by-2 transition matrix written as :

$$\mathrm{P}_{\mathrm{corr}} = \begin{bmatrix} p^c_{11}(t) & 1 - p^c_{22}(t) \\ 1 - p^c_{11}(t) & p^c_{22}(t) \end{bmatrix} \tag{4.13}$$

whose elements can be both static and time-varying. In that case, this is the same specification as defined in the equation (4.10). Given the transition matrices of the first and the second levels, the partition of the space is represented by transition matrix P expressed as :

$$\mathrm{P} = \mathrm{P}_{\mathrm{vol}} \otimes \mathrm{P}_{\mathrm{corr}} \tag{4.14}$$

of size $2^{K+1} \times 2^{K+1}$.

Given this space partition the first and the second levels are then linked. This relation is fully probabilistic and attributes a weight to the decision related to the correlation given the decision of the univariate volatility. The relation has been obtained with the use of a *coupling* matrix. Coupling matrices are generally used in the context of coupled HMMs where a set of Markov chains influence each other. In our specific case, the matrices used conform to the Markov property. Thus, the dynamic of the junction between the two levels is effected with the use of an abstract Markov chain. This linking relation does not emit directly observation but plays the part of attributing a weight between volatility and correlations. This coupling matrix is of size 2-by2 and is written as :

$$\mathrm{P}_{\text{coupl}} = \begin{bmatrix} c_{11}(t) & 1 - c_{22}(t) \\ 1 - c_{11}(t) & c_{22}(t) \end{bmatrix} \tag{4.15}$$

The elements of this matrix can also be time-varying given the input and are defined with the logistic specification of the equation (4.10).

To summarize the foregoing information on the probabilistic decision tree : the entire ordered hierarchy is based on a set of HMMs which partition the space. While the partitioning of space, as usually considered in binary tree models, is a recursive process following binary rules, our model has the sort which is static and defined a priori. The decision process occurs according to a cascade of ordered synchronous HMMs. The coupling process lacks the Markovian property, but can be treated as a vector of probabilities whose sum is equal to unity. By establishing a link between the HMMs, the coupling process captures the inter-process influences. Conditional time-varying transitions and coupled probabilities allow the structure to adapt for each observation.

Our model uses a set of HMMs to counterbalance the influence of the univariate volatility on the correlations. The first level of the model discriminates between low-volatiliy series and high-volatility series each time, while the second level orders the correlations on a scale of high or low. The input variable can be either endogenous or exogenous.

Specification for univariate volatilities

In our decision tree, the first level distinguishes between low and high volatility. Thus, this decision step parametrizes a univariate regime switching GARCH model. Within the literature on time series, many specifications have been proposed. Most of them have the drawback that they require approximation schemes to avoid the path dependency problem. This is particularly the case with the models suggested by Gray (1996) and Klaassen (2002). Apart from this numerical aspect, specifications with approximations can entail difficulties in the interpretation of the processes corresponding to each regime. These considerations led us to use the Markov-switching GARCH proposed by Haas, Mittnik, and Paollela (2004b). This model is expressed as follows :

$$\begin{pmatrix} h_{1,t} \\ \vdots \\ h_{N,t} \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_N \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{pmatrix} y_{t-1}^2 + \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_N \end{pmatrix} \odot \begin{pmatrix} h_{1,t-1} \\ \vdots \\ h_{N,t-1} \end{pmatrix} \tag{4.16}$$

where $\odot$ stands for the element-by-element multiplication. The stationarity condition implies $\alpha_n + \beta_n < 1$ for each $n = 1, ..., N$. Besides its computational advantages, this model has a clear cut interpretation. It assumes that the conditional variance can switch between N separate GARCH models that evolve in parallel. Because we need two regimes, each time series involves eight parameters.

Specification for correlations

The second level discriminates between low and high conditional correlations. As we did with the first level, we use a specification which does not require approximation in order to facilitate the interpretation of the regimes. Haas and Mittnik (2008) have proposed an extension of the Markov-switching GARCH to the multivariate case. Since the conditional covariances of the standardized residuals has a BEKK dynamic, we can use this Markov-switching extension in the scalar case. It can be written as :

$$\begin{pmatrix} Q_{1,t} \\ \vdots \\ Q_{N,t} \end{pmatrix} = \begin{pmatrix} \Omega_1 \\ \vdots \\ \Omega_N \end{pmatrix} + \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \varepsilon_{t-1}\varepsilon'_{t-1} + \begin{pmatrix} b_1 \\ \vdots \\ b_N \end{pmatrix} \odot \begin{pmatrix} Q_{1,t-1} \\ \vdots \\ Q_{N,t-1} \end{pmatrix} \tag{4.17}$$

Stationarity conditions imply the intercept Ω_n to be a positive definite matrix and $a_n + b_n < 1$ for $n = 1, \ldots N$. This specification requires $K(K+1)/2+4$ parameters.

4.2.3 Estimation

Estimation of the model is done with maximum likelihood. One of the advantages of the DCC of Engle (2002) is the possibility to estimate the model in a two steps approach. The way to do this would be to estimate the parameters related to the univariate volatilities in the first step and in the second step the parameters related to the correlations. In our case, because of the relation between the degarching process and the correlation, we must estimate the model in one step. With the assumption of normality, the log-likelihood can be written :

$$L = -\frac{1}{2}\sum_{t=1}^{T}\left(K\log(2\pi) + \log(|H_t|) + y'_t H_t^{-1} y_t\right) \tag{4.18}$$

To be able to maximize the likelihood we need to make inferences on the state of the various Markov chains of the model. The strategy developed in our paper is to convert the complex dynamic of the factorial and coupled HMM into a simple regular HMM so that we can use standard tools for filtering and smoothing probabilities. A conversion relationship between coupled HMM and standard HMM has been proposed by Brand (1997). Given $\mathrm{P}_{\mathrm{S|S}}$ and $\mathrm{P}_{\mathrm{S'|S'}}$ two transition matrices, $\mathrm{P}_{\mathrm{S|S'}}$ and $\mathrm{P}_{\mathrm{S'|S}}$ two coupling matrix, the relationship of Brand (1997) is given by :

$$(\mathrm{P}_{\mathrm{S|S}} \otimes \mathrm{P}_{\mathrm{S'|S'}}).R(\mathrm{P}_{\mathrm{S'|S}} \otimes \mathrm{P}_{\mathrm{S|S'}}) \tag{4.19}$$

where R is a row permute operator swapping fast and slow indices. In our case, because we use only one coupling vector in a downward direction, the regular HMM representation $\mathrm{P}_{\mathrm{reg}}$ of our model can be expressed as :

$$\mathrm{P}_{\mathrm{reg}} = (\mathrm{P}_{\mathrm{vol}} \otimes \mathrm{P}_{\mathrm{corr}}) \odot (\mathrm{P}_{\mathrm{coupl}} \otimes (\iota\iota')) \tag{4.20}$$

with ι a vector of ones of length 2^{K+1}.
The conversion into regular HMM allows the use of standard tools for the inference of the Markov chains. Let ξ_{jt} the probability to be in regime j given the information set available at time $t-1$ and η_{jt} the density under the regime j. The probability to be in each regime at time t given the observations set up to t, written $\hat{\xi}_{t|t}$, can be calculated using the following expression of the so-called Hamilton's filter :

$$\hat{\xi}_{t|t} = \frac{(\hat{\xi}_{t|t-1} \odot \eta_t)}{\mathbf{1}'(\hat{\xi}_{t|t-1} \odot \eta_t)} \tag{4.21}$$

and :

$$\hat{\xi}_{t|t+1} = \mathrm{P}_{\mathrm{reg}} \times \hat{\xi}_{t|t} \tag{4.22}$$

where $\odot$ denotes element-by-element multiplication. The conversion to regular HMM has the advantage that it makes for ease in the tasks of computation and interpretation. Each state of the regular representation corresponds to a combination of possible cases, e.g. the first series in a high or a low volatility period, the second in high or low, and so on, and similarly for the correlations. The main drawback of the conversion method lies in the complexity of the maximum likelihood estimation. Optimization methods based on gradient methods using numerical derivatives can encounter difficulties with local maxima. It is therefore advisable to test several initial conditions.

4.3 Application

For the first application, we propose to illustrate our proposed model on a daily dataset. We consider futures prices of 10-year Treasury Bonds and the S&P 500 from September 1994 to February 2003, a total of 2201 observations. The two series were sourced from DataStream, and are plotted in figure 4.4. Returns have been calculated as 100 times the absolute difference of the logarithm of each series minus the sample mean. Descriptive statistics of the log returns are shown in table 4.1. All the results reported in the analysis were generated with Matlab on Linux.

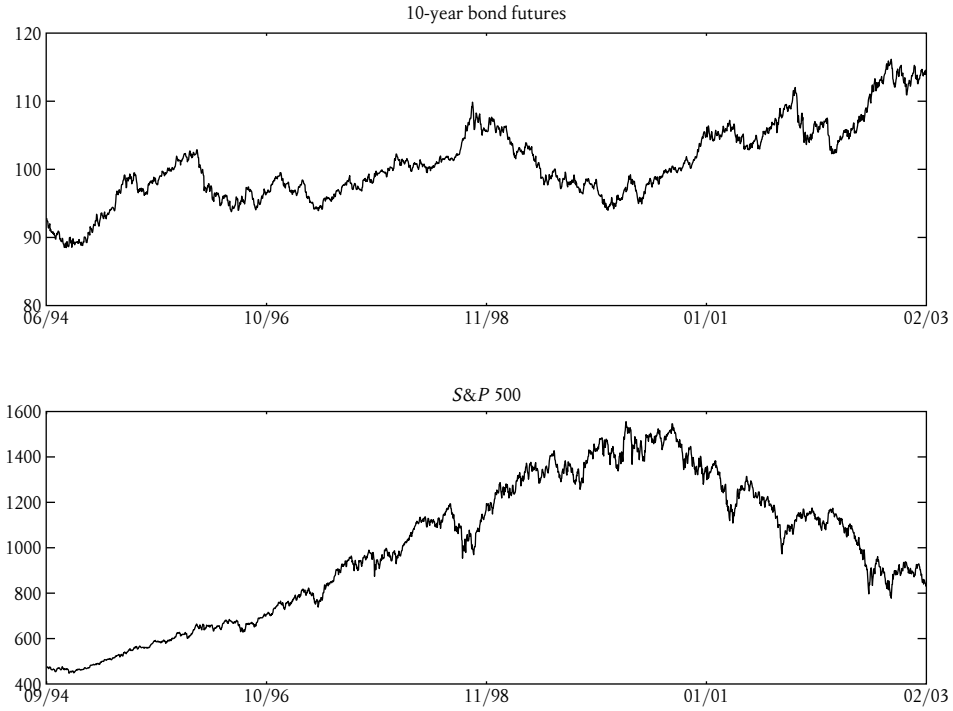


FIGURE 4.4 – Futures prices of 10-year bond and S&P500 1994 to July 2003.

With two series, the dynamic of the univariate volatility level can be represented by a 4-by-4 transition matrix (see equation (4.12)). The second level has a 2-by-2 transition matrix. Thus, given equation (4.14), the space is partitioned into eight sub-spaces, each corresponding to a special case. Let $h_{t,Y}$ and $h_{t,P}$ the univariate volatilities of the bonds and the S&P 500. Here, the superscript h denotes a high volatility regime, and the superscript l a low volatility regime. A high regime of

	Bond $	S&P500
Minimum	-2.2154	-7.7621
Maximum	1.4175	5.7549
Mean	0.0094	0.0258
Variance	0.1430	1.5191
Skewness	-0.5310	-0.1236
Kurtosis	5.4602	6.5324
Jarque-Bera	658.2178 (1.0000e−03)	1.1494e+03 (1.0000e−03)
Box-Ljung Q(25)	30.9139 (0.1919)	29.8141 (0.2313)
Box-Ljung Q(50)	65.4385 (0.0703)	64.6047 (0.0802)

TABLE 4.1 – Descriptive statistics for the 10-year Treasury Bond Futures and the S&P 500 Futures.

correlations is written by R_t^h and a low by R_t^l. The enumeration of the various cases corresponds to :

1. $\{h_{t,Y}^h, h_{t,P}^h\}, \{R_t^h\}$
2. $\{h_{t,Y}^h, h_{t,P}^h\}, \{R_t^l\}$
3. $\{h_{t,Y}^h, h_{t,P}^l\}, \{R_t^h\}$
4. $\{h_{t,Y}^h, h_{t,P}^l\}, \{R_t^l\}$
5. $\{h_{t,Y}^l, h_{t,P}^h\}, \{R_t^h\}$
6. $\{h_{t,Y}^l, h_{t,P}^h\}, \{R_t^l\}$
7. $\{h_{t,Y}^l, h_{t,P}^l\}, \{R_t^h\}$
8. $\{h_{t,Y}^l, h_{t,P}^l\}, \{R_t^l\}$

With two series, the decision tree can be represented as in figure 4.5. The coupling matrix is then of size 2×2. This coupling matrix links each state $\{h_{t,Y}^i, h_{t,P}^j\}$, $i, j = h, l$ with a state $\{R_t^i\}$, $i = h, l$. The relation introduced by this vector can be viewed as a weighted function.
With two series, the number of parameters related to the first level is 8 and 12 for the correlations. With 2 parameters for the coupling matrix, the total number of parameters to estimate is 30. As explained in the previous section, all first-level decisions can be summarized in a single transition matrix, which is the cross product of the two transition matrices related to each series.

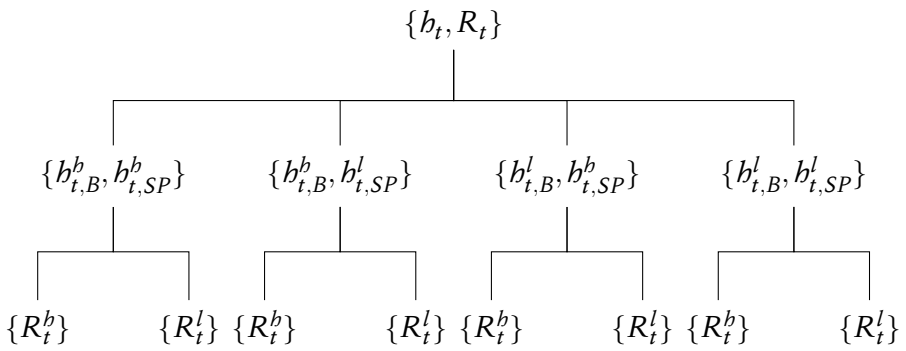


FIGURE 4.5 – Structure of the decision process for a bivariate dataset.

First level

	10-year bond futures		S&P500	
Parameters	Estimate	standard errors	Estimate	standard errors
ω_1	0.0027	4.07e-6	0.0082	5.50e-6
α_1	0.0434	1.66e-4	0.0694	1.92e-4
β_1	0.9394	5.45e-4	0.9263	1.91e-4
ω_2	0.0342	0.0101	0.0157	6.48e-5
α_2	0.1652	0.3214	0.0911	0.0013
β_2	0.8201	0.3342	0.8992	0.001
p_{11}	0.9981	1.50e-4	0.9998	5.08
p_{22}	0.0010	7.69e-8	0.6522	0.0154

Coupling matrix

Parameters	Estimate	standard errors
p_{11}	0.2522	0.0584
p_{22}	0.8629	0.0049

Second level

Parameters	Estimate	standard errors
a_1	0.1532	0.1372
b_1	0.7349	0.2126
const	$\begin{bmatrix} 0.1281 & 0.2852 \\ & 0.6880 \end{bmatrix}$	$\begin{bmatrix} 8.44e-4 & 0.0013 \\ & 0.0092 \end{bmatrix}$
a_2	0.0138	2.76e-5
b_2	0.9567	2.61e-4
const	$\begin{bmatrix} 0.0037 & -0.0087 \\ & 0.0164 \end{bmatrix}$	$\begin{bmatrix} 5.32e-6 & 1.53e-5 \\ & 0.0037 \end{bmatrix}$
p_{11}	0.8843	0.0154
p_{22}	0.2502	0.0305

TABLE 4.2 – Model estimates.

4.3.1 Results of the model

We apply our model with constant transition probabilities. This configuration has no input. Estimated correlations are plotted on figure 4.6(a). Our results confirm those of Colacito and Engle (2006), who had previously studied this dataset. Correlations are positive in the first part of the sample and become negative, and remain negative, from halfway on to the end.

Figure 4.6(b) shows the smoothed probabilities selected by the decision process. While the deterministic decision tree attributes a model for each partition of the space, each partition being represented thanks to the threshold classification followed, our model discriminates between regimes. Consequently, the smoothed probabilities of figure 4.6(b) represent the combination of regimes that fit the data. Interpretation is anything but obvious in the former case.
The model clearly identifies three combinations of regimes. The identification of the nature of regimes can be done using the table 4.2 of estimated parameters. We first note that the 10-year bond futures serie does not seem to have a change of regime. The following interpretation of theses combinations are, by chronological order :

1. $\{h^2_{t,Y}, h^1_{t,P}, R^1_t\}$: normal volatility for bond, low volatility for S&P500 and positive correlations regime.
2. $\{h^2_{t,Y}, h^2_{t,P}, R^1_t\}$: normal volatility for bond, high volatility for S&P500 and positive correlations regime.
3. $\{h^2_{t,Y}, h^1_{t,P}, R^2_t\}$: normal volatility for bond, low volatility for S&P500 and negative correlations regime.

The results of the estimated transition probability matrices clearly shows that the first regime are persistent for the bond. Our model does not identify low and high volatility period. This is not the case of the index where the model identifies two regimes. This is also the same for the correlations and the first part of the sample is clearly a regime of positive correlations while and the second part has negative correlations.
The estimated results of the coupling transition matrix gives us the relationship between the first and the second level. This 2-by-2 transition matrix with constant probabilities is written as :

$$\mathrm{P}_{\text{coupl}} = \begin{bmatrix} c_{11} & 1-c_{22} \\ 1-c_{11} & c_{22} \end{bmatrix} \tag{4.23}$$

where the first state links the volatility level with the positive correlation regime and the second rely the first level with the negative correlation regime. Estimated parameters are :

$$\widehat{\mathrm{P}}_{\text{coupl}} = \begin{bmatrix} 0.2522 & 0.1371 \\ 0.7478 & 0.8629 \end{bmatrix} \tag{4.24}$$

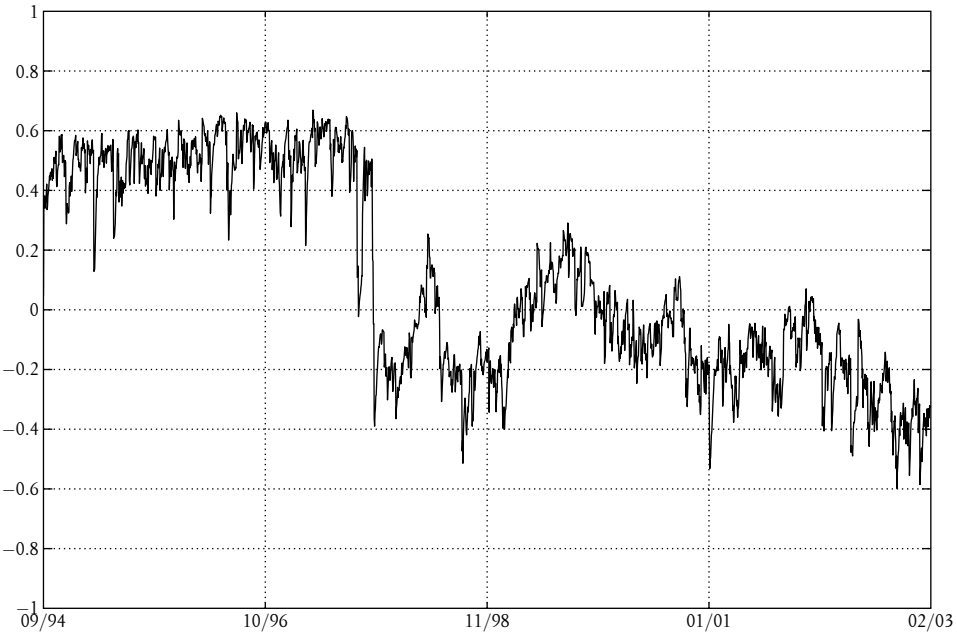
This result means that in general, volatility is associated with negative correlations.

4.3.2 Deepening of the relationship between first and second level

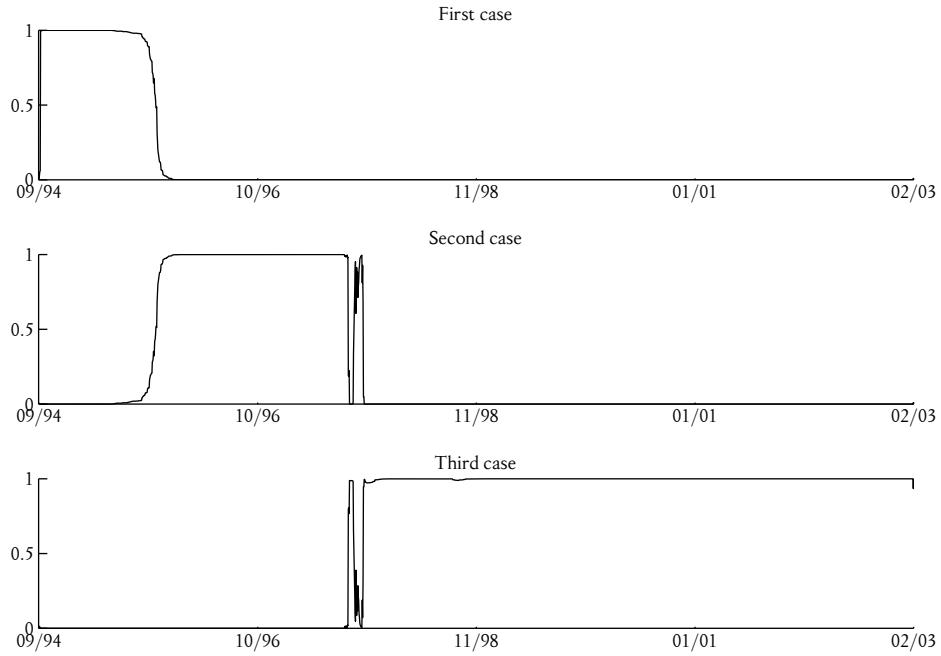
In the later application, we use a very simple transition matrix to link the first and the second level. This specification has the advantage of being attractive from a computational point of view but seems to be quite restrictive. Indeed the linking relation is common to all case of the first stage. In this subsection, we extend the later specification by introducing a specific relation for all the possible cases at the first level. In that case, there are four two-by-two coupling matrix matrix P^i, $i = 1, ..., 4$. Each of these links a pair $\{h_{t,B}, h_{t,SP}\}$ with a case $\{R_t\}$. The decision of this extended case can be summarized as in figure 4.7.
While the simple model needs only two parameters for the coupling matrix, the extended case needs eight parameters. Estimation is done using maximum likelihood using a regular HMM representation. Estimated correlations are plotted in figure 4.8(a) and the smoothed probabilities of the relevant regimes are plotted in figure 4.8(b) ; estimated parameters are in table 4.3.

Comparing to the previous specification, the introduction of specific coupling matrices increases the explanatory power of the model. Estimation of the model exhibits six cases :

1. $\{h^1_{t,Y}, h^1_{t,P}, R^1_t\}$: low volatility for bond, low volatility for S&P500 and negative correlations regime.
2. $\{h^1_{t,Y}, h^1_{t,P}, R^2_t\}$: low volatility for bond, low volatility for S&P500 and positive correlations regime.
3. $\{h^1_{t,Y}, h^2_{t,P}, R^2_t\}$: low volatility for bond, high volatility for S&P500 and positive correlations regime.
4. $\{h^2_{t,Y}, h^1_{t,P}, R^1_t\}$: high volatility for bond, low volatility for S&P500 and negative correlations regime.



(a) ESTIMATED CORRELATIONS.



(b) SMOOTHED PROBABILITIES.

FIGURE 4.6 – Estimated correlations and smoothed probabilities.

5. $\{h^2_{t,Y}, h^1_{t,P}, R^2_t\}$: high volatility for bond, low volatility for S&P500 and positive correlations regime.
6. $\{h^2_{t,Y}, h^2_{t,P}, R^2_t\}$: high volatility for bond, high volatility for S&P500 and positive correlations.

While the previous specification only shows us that in general volatility is associated with negative correlations, our extended model gives a more precise information about the relationship between volatility and correlations. Given the estimated parameters of the four coupling matrix, we can deduce that there is a strong probability to have positive correlation when bond and S&P500 are in low volatility regime and when bond has low volatility and S&P500 high has high volatility. On

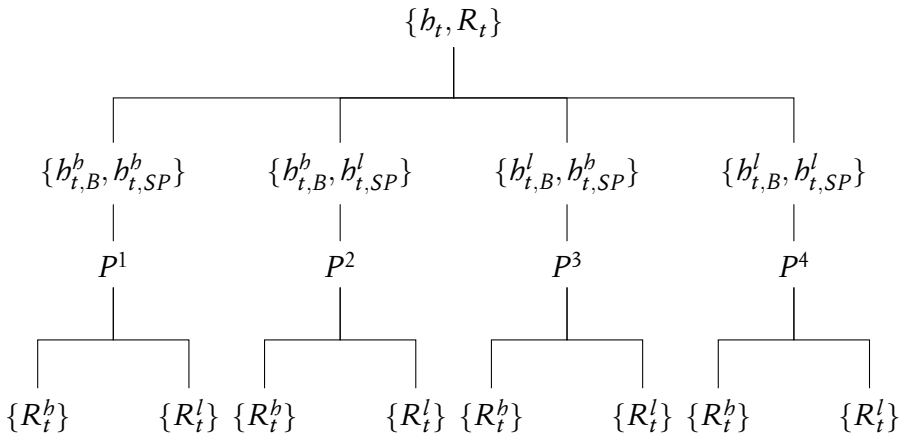
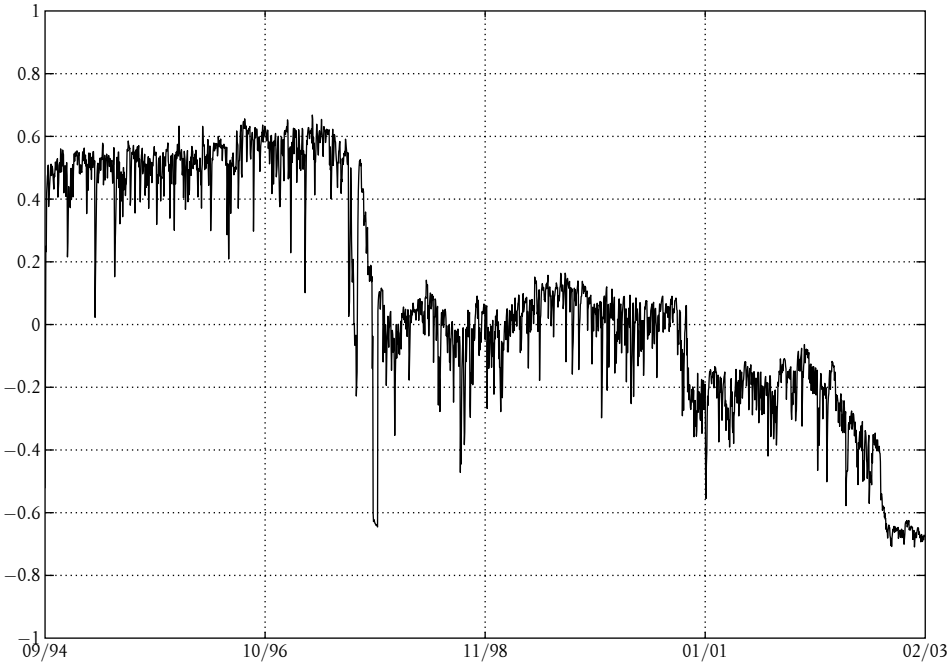


FIGURE 4.7 – Structure of the decision process for a bivariate dataset of the extended version.

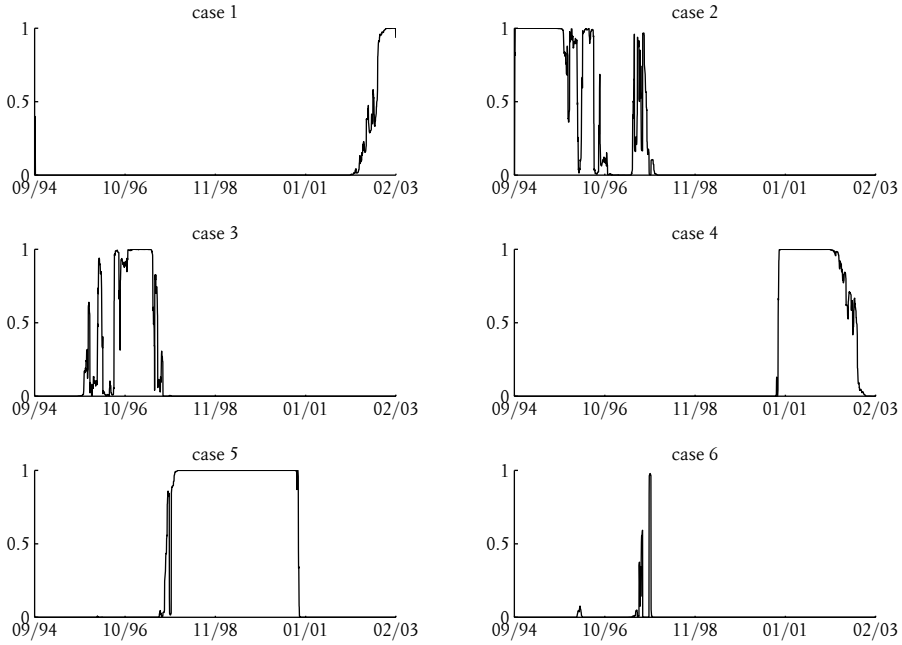
the other hand, negative correlations seems to be associated with a high volatility of the bond. This particularity is more significative when the S&P500 is in low volatility regime.

4.4 Conclusion

The model we have presented in this paper has a natural and easy interpretation. Based on the Hidden Markov Decision Tree introduced by Jordan, Ghahramani, and Saul (1997), we have developed a stochastic decision tree linking the dynamics of univariate volatility with the dynamics of the correlations. Unlike conventional approaches based on deterministic decision trees, our model allows a probabilistic point of view of the relationship between univariate volatility and correlations. We have restricted ourselves the complexity of the tree-structure and more refined architecture remains to be done. It also remains to develop estimation tools more suitable when the dimension of the problem increases.



(a) ESTIMATED CORRELATIONS.



(b) SMOOTHED PROBABILITIES.

FIGURE 4.8 – Estimated correlations and smoothed probabilities.

First level

	10-year bond futures		S&P500	
Parameters	Estimate	standard errors	Estimate	standard errors
ω_1	0.0017	1.03e-06	0.0115	0.00015
α_1	0.0406	0.00029	0.0867	0.00129
β_1	0.9488	7.55e-05	0.8391	0.00933
ω_2	0.0137	6.35e-05	0.0831	0.00013
α_2	0.0706	0.00050	0.3582	1.11e-05
β_2	0.9195	0.00078	0.6417	1.06e-05
p_{11}	0.9987	0.00227	0.9999	1.02e-28
p_{22}	0.0010	6.272e-07	0.0001	1.70e-07

Coupling matrix

Parameters	Estimate	standard errors
p_{11}^1	0.7558	0,002485
p_{22}^1	0.09319	1,28e-05
p_{11}^2	0.7661	0,00066
p_{22}^2	0.11846	1,66e-05
p_{11}^3	0.2874	0,00045
p_{22}^3	0.0267	8,71e-07
p_{11}^4	0.5686	0,00026
p_{22}^4	1.00e-05	6,43e-08

Second level

Parameters	Estimate	standard errors
a_1	8.8334e-04	8.473e-06
b_1	0.9716	0.00086
const	$\begin{bmatrix} 0.0008 & -0.0031 \\ & 0.0127 \end{bmatrix}$	$\begin{bmatrix} 1.12e-05 & 0.00014 \\ & 0,00211 \end{bmatrix}$
a_2	0.0344	0.00044
b_2	0.4325	0.04093
const	$\begin{bmatrix} 0.0834 & 0.1107 \\ & 0.1836 \end{bmatrix}$	$\begin{bmatrix} 0,00111 & 0,01074 \\ & 0,11735 \end{bmatrix}$
p_{11}	0.8223	0.00144
p_{22}	0.9714	6.76e-06

TABLE 4.3 – Model estimates.

CHAPITRE

5

Conclusions et Perspectives.

L'étude des propriétés statistiques observées sur les séries financières a engendré une littérature volumineuse. Plusieurs classes de modèles ont ainsi émergé au fil des années (GARCH, volatilité stochastique, multifractal) afin de modéliser l'évolution des rendements. Plus récemment, l'intérêt des chercheurs s'est porté sur l'étude de la volatilité des séries vectorielles de rendements afin de mieux comprendre les liens reliant diverses séries de rendements.

Les modèles GARCH multivariés à corrélations conditionnelles, développés avec l'article de Bollerslev (1990) et le modèle CCC, constituent un cadre d'analyse des variances et corrélations conditionnelles des séries financières. Cependant, si la formulation originale du CCC a constitué une avancée majeure dans la modélisation des covariances conditionnelles, son hypothèse de corrélations conditionnelles constantes dans le temps s'est avérée très réductrice empiriquement. L'introduction de corrélations conditionnelles variant dans le temps est proposée presque simultanément par Tse and Tsui (2002) et Engle and Sheppard (2001) (modèles DCC). Les modèles à corrélations conditionnelles dynamiques constituent désormais un axe de recherche très actif de l'économétrie financière.

Dans cette thèse, nous nous sommes focalisés sur les modèles DCC à changements de régime. Ces modèles sont un outil précieux pour les praticiens afin de comprendre la dynamique de séries financières présentant des ruptures structurelles. Plusieurs approches classiques ont déjà été proposées dans la littérature, basées sur un mécanisme de saut qui peut être déterministe ou stochastique.

Notre travail s'intéresse à l'écriture du mécanisme permettant de passer d'un régime à l'autre en se focalisant sur les mécanismes probabilistiques. L'approche classique en économétrie consiste à supposer que le mécanisme de passage d'un régime à un autre est déterminé selon une variable aléatoire cachée suivant une chaîne de Markov au premier ordre. Cette modélisation par modèle de Markov caché (HMM) a déjà été appliquée à diverses extensions du modèle DCC. Les contributions de cette thèse ont pour point commun d'être toutes basées sur des extensions du modèle HMM de base. Ses extensions ont été développées dans une autre discipline que l'économétrie financière : les modèles graphiques probabilistes (MGP). Les MGP sont une formulation hybride combinant la théorie des graphes et celle des probabilités dont les applications concernent la reconnaissance d'écriture, de la parole etc. Ainsi, nous proposons des modèles DCC à changements de régime dans lesquels nous avons substitué au modèle HMM classique des extensions issues du formalisme des MGP.

Dans le chapitre 2 nous proposons un cas particulier du modèle RSDC de Pelletier (2006) basé sur le modèle HMM hiérarchique de Fine, Singer, and Tishby (1998). Le modèle RSDC suppose que les corrélations évoluent entre des matrices de corrélations constantes dans le temps, dont le passage de l'une à l'autre s'effectue via une chaîne de Markov cachée. Dans notre modèle, la chaîne de Markov est remplacée par une structure hiérarchique Markovienne cachée. Ainsi, dans le HRSDC à deux régimes, les deux régimes principaux sont une combinaison de sous régimes. La matrice de corrélation conditionnelle évolue alors entre quatre matrices constantes dans le temps. Cette spécification apporte une meilleurs définition des régimes en augmentant leur granularité.

Le modèle du chapitre 3 utilise une autre extension du modèle HMM afin d'apporter davantage de flexibilité à l'approche classique. En effet, dans un modèle DCC avec changements de régime selon une chaîne de Markov cachée classique (comme le Markov-switching), tous les éléments de la matrice des corrélations conditionnelles ont la même dynamique de saut. Cette hypothèse peut s'avérer restrictive pour l'étude de séries financières en période de crise puisque cette spécification impose un changement de régime commun à toutes les corrélations. Pour relaxer cette hypothèse, nous utilisons le modèle HMM factorisé (FHMM) de Ghahramani and Jordan (1997) dans lequel la variable d'état est décomposée en plusieurs variables d'état. Il y a donc plusieurs chaînes de Markov en parallèle et l'état du modèle est déterminé par la combinaison de ces chaînes. A la différence du modèle HMM classique, cette approche permet d'attribuer une dynamique de saut à

chaque élément de la matrice des corrélations.
La contribution du chapitre 4 développe un arbre de décision afin de relier le niveau des volatilités individuelles aux corrélations. L'approche usuelle des arbres de décision est généralement construite dans un cadre déterministe. Les décisions sont prises de façon binaire et l'arbre se construit selon une procédure séquentielle. L'arbre de décision Markovien caché que nous utilisons a été proposé par Jordan, Ghahramani, and Saul (1997). Il se situe dans un cadre purement probabiliste et peut s'interprêter comme un HMM à la fois factorisé et couplé. Appliqué au modèle DCC de Engle and Sheppard (2001), le premier niveau de l'arbre discrimine entre un régime faible et fort de volatilité, tandis que le second niveau discrimine entre un régime faible et fort des corrélations.

Ainsi, les contributions de cette thèse se situent à l'interface de deux disciplines : l'économétrie financière et les modèles graphiques probabilistes. Les résultats obtenus montrent que les extensions HMM provenant des MGP peuvent s'appliquer facilement aux modèles à changements de régime utilisés en économie. Ils apportent par ailleurs un gain en terme de pouvoir explicatif pour le praticien. Cependant, l'introduction des extensions laisse beaucoup de questions sans réponses. Tout d'abord, au niveau théorique, les propriétés probabilistes des extensions du modèle HMM restent à confirmer. S'il existe en effet de nombreuses études dans le cadre du modèle standard HMM, il n'y a, à notre connaissance, pas d'analyses précises du comportement asymptotique des chaînes de Markov structurées. Cette lacune théorique rend également délicate la recherche des propriétés asymptotiques des modèles de type DCC construits à partir d'extensions de HMM.
Par ailleurs, dans un registre plus pratique, il apparaît nécessaire de développer des méthodes d'estimation adaptées à ces processus. Dans cette thèse, nous avons fait le choix de convertir les HMM structurés en HMM de base. Cette stratégie avait pour but de permettre l'utilisation d'outils classiques d'inférence et d'estimation de l'économétrie financière. Ainsi, les modèles présentés dans cette thèse s'estiment par maximum de vraisemblance et l'inférence sur l'état de la chaîne de Markov est réalisée avec le filtre d'Hamilton. Néanmoins la stratégie consistant à convertir une structure Markovienne structurée en écriture HMM équivalente présente une malédiction de la dimension. En effet, les représentations équivalentes HMM présentent alors généralement une matrice de transition de grande taille.

Au final, les résultats présentés dans cette thèse montrent que les extensions HMM provenant des modèles graphiques probabilistes permettent d'améliorer le mécanisme de changement de régime par rapport à l'approche Markov-switching classique. Si dans cette thèse nous nous sommes restreint à l'étude des corrélations de séries financières, ces extensions trouveront aussi leur utilité pour raffiner les approches de type Markov-switching VAR, mais aussi pour l'analyse de données hautes fréquences.

ANNEXE

A

A.1 An example of HHMM

Figure A.1 is an example of HHMM with four levels and six emitting states.

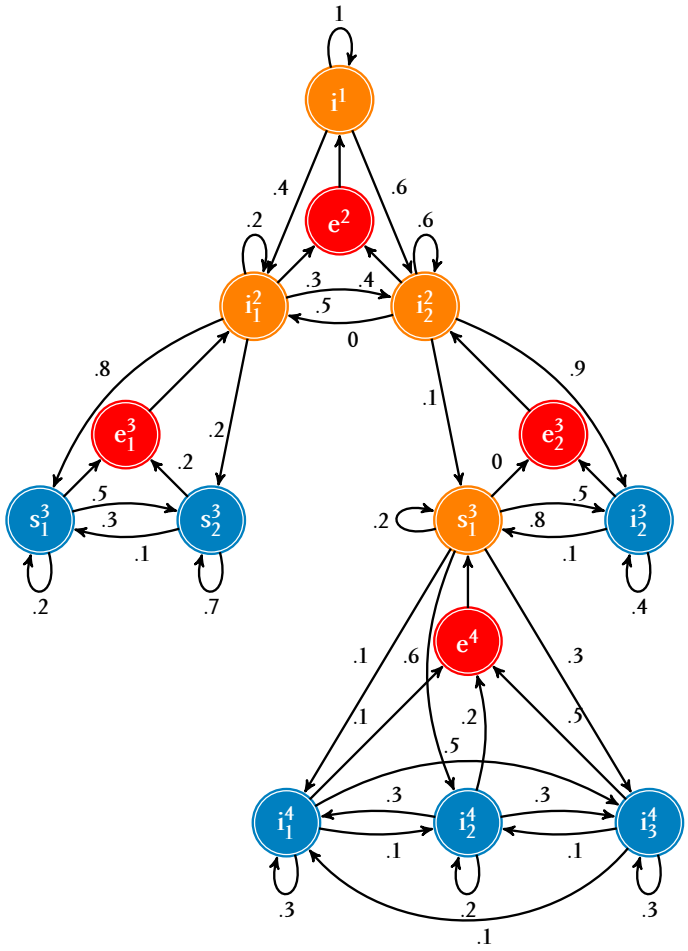


FIGURE A.1 – An example of Hierarchical Hidden structure.

A.2 EM algorithm

The origin of the Expectation-Maximization (EM) algorithm comes from the difficulties to estimate the parameters of models based on unobserved latent variables. More specially, it is an alternate approach face to usual iterative method to reach the optimum of an objective function. In the case of HMM/Markov-Switching models, the difficulties comes generally from two levels : (i) the likelihood function has numerous optimums (ii) optimizing the likelihood function is analytically intractable.
The firsts major contributions to elaborate this technique are due to Baum and T. (1966); Baum, T., Soules, and Weiss (1970). They developed the two steps methodology which consist to first compute the expectation of the likelihood function with the latent variables (E step) and then maximizing this expected likelihood (M step). From a theoretical point if view, the condition of convergence of the algorithm were proved by Dempster, Laird, and Rubin (1977) and improved few years later by Wu (1983). The literature on the subject was later expanded considerably and

many variants have emerged (see for example Meng and van Dyk (1997) and also Geoffrey and Thriyambakam (1997)).
More formally, the mechanic of the EM algorithm is to proceed by successive maximization based on observable data and information that we have every iterations on the latent variables. Instead of trying to directly calculate the conditional expectation $\mathbb{E}[\log f(\varepsilon, s|\theta)|\varepsilon, \theta]$, which is impossible because of s, the problem is being circumvented by focusing on the calculation of the quantity :

$$\mathscr{Q}(\theta|\theta_p) \overset{def}{=} \mathbb{E}_{\theta_p}[\log f(\varepsilon_{1:T}, s_{1:T}; \theta')|\varepsilon_{1:T}]$$

that it is calculable. Because the log-likelihood of the complete data is convex, it is possible to use Jensen inequality to deduce that, for all $(\theta', \theta) : \mathscr{Q}(\theta'|\theta) \geq \mathscr{Q}(\theta|\theta) \Longrightarrow \ell(\theta') \geq \ell(\theta)$. The two steps methodology for the p^{th} is then as follow :

$$\begin{aligned} &① \textit{ E-step} : \mathscr{Q}(\theta|\theta_p) = \mathbb{E}_{\theta_p}[\log f(\varepsilon, s; \theta')|\varepsilon] \\ &② \textit{ M-step} : \hat{\theta}_{p+1} = \arg\max_{\theta} \mathscr{Q}(\theta|\theta_p) \end{aligned}$$

By iterating ① and ②, the sequence $(\theta_p)_{p\in\mathbb{N}}$ ensures the convergence of the sequence $(\ell(\theta_p))_{p\in\mathbb{N}}$ to the true vector of parameters.

A.3 Scaling factor

Computing the quantities $\alpha_t(q)$ and $\beta_t(q)$ can lead to underflow problem. A easy way to circumvent this issue is to introduce a *scaling factor*, written as ς. In that case, initialization of Forward variable, written $\breve{\alpha}_t(q)$ is done by taking $\varsigma(1) = \sum_q \alpha_1(q)$:

$$\breve{\alpha}_1(q) = \frac{\alpha_1(q)}{\varsigma(1)}$$

We can then compute the *intermediary Forward variable*, written $\bar{\alpha}_{t+1}(q)$:

$$\bar{\alpha}_{t+1}(q) = f_q(\varepsilon_{t+1}) \sum_{q'} \sum_{d} \breve{\alpha}_t(q_t) \tilde{a}^d(q', q)$$

which lead to the scaling factor $\varsigma(t+1)$:

$$\varsigma(t+1) = \sum_q \bar{\alpha}_{t+1}(q)$$

The adjusted Forward variable can then be written as follow :

$$\breve{\alpha}_{t+1}(q) = \frac{\bar{\alpha}_{t+1}(q)}{\varsigma(t+1)}$$

Similar calculus can be applied to obtain the Backward variable. By initializing with $\breve{\beta}_T(q) = 1$, the intermediary quantity is expressed as :

$$\bar{\beta}_t(q) = \sum_{q_t} \sum_{d} \breve{\beta}_{t+1}(q') f_{q'}(\mathbf{y}_{t+1}) \tilde{a}^d(q, q')$$

and the adjusted Backward variable is obtained by computing for each $t = T-1, ..., 1$:

$$\breve{\beta}_t(q) = \frac{\bar{\beta}_t(q)}{\varsigma(t)}$$

The two quantities $\breve{\alpha}_t(q)$ and $\breve{\beta}_t(q)$ allow to rewriting the probabilities $\gamma_t^p(q)$ and $\xi_t^p(q', q, d)$ as :

$$\begin{aligned} \gamma_t^p(q) &= \breve{\alpha}_t(k) \breve{\beta}_t(k) \\ \xi_t^p(q', q, d) &= \breve{\alpha}_t(q') \tilde{a}^d(q', q) f_{q'}(\mathbf{y}_{t+1}) \breve{\beta}_{t+1}(q) \end{aligned}$$

A.4 Exchange rate data

The exchange rate database is that of Harvey, Ruiz, and Shephard (1994) and was also used by Kim, Shephard, and Chib (1998) and Pelletier (2006). It is freely downloadable at the url :

`http://www.nuffield.ox.ac.uk/users/shephard/pub.aspx`

It contains week-days close exchange rates against US dollar for Pound, Deutschmark, Yen and Swiss-Franc over the period 1/10/81 to 28/6/85. This series are plotted on figure A.2.

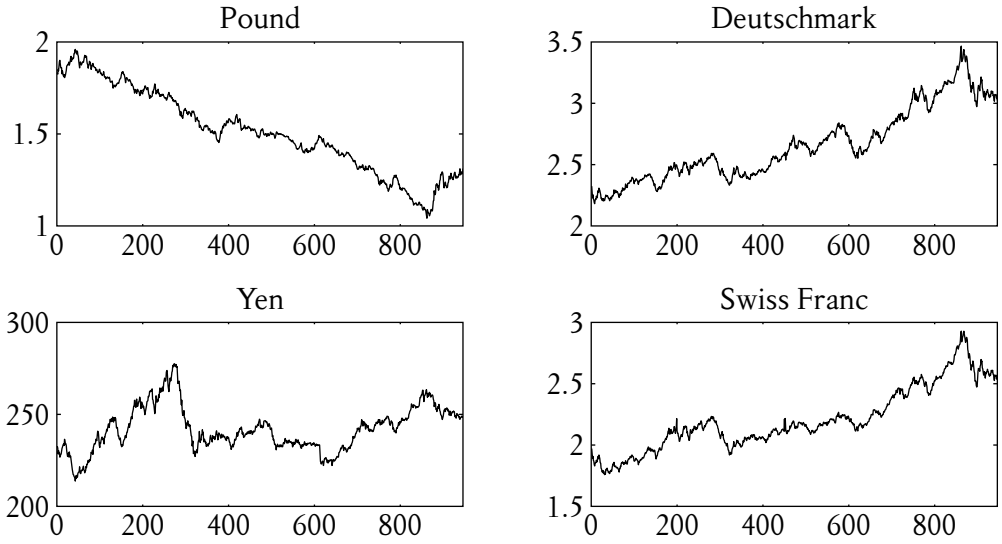


FIGURE A.2 – Exchange rate database.

A.5 S&P500 index futures and 10-year bond futures

This database is that of Colacito and Engle (2006) (plotted on figure A.3) and available at the url :

`http://www.unc.edu/~colacitr/Research/Files/EC2006.zip`

It contains daily returns of S&P500 index futures and 10-year bond futures from January 1990 to August 2003 (3912 observations).

A.6 Estimated parameters of the HRSDC for the real database applications

Table A.2 shows the parameters estimations of the correlations process for the database of exchange rate data ; table A.1 for the database of Engle and Colacito.

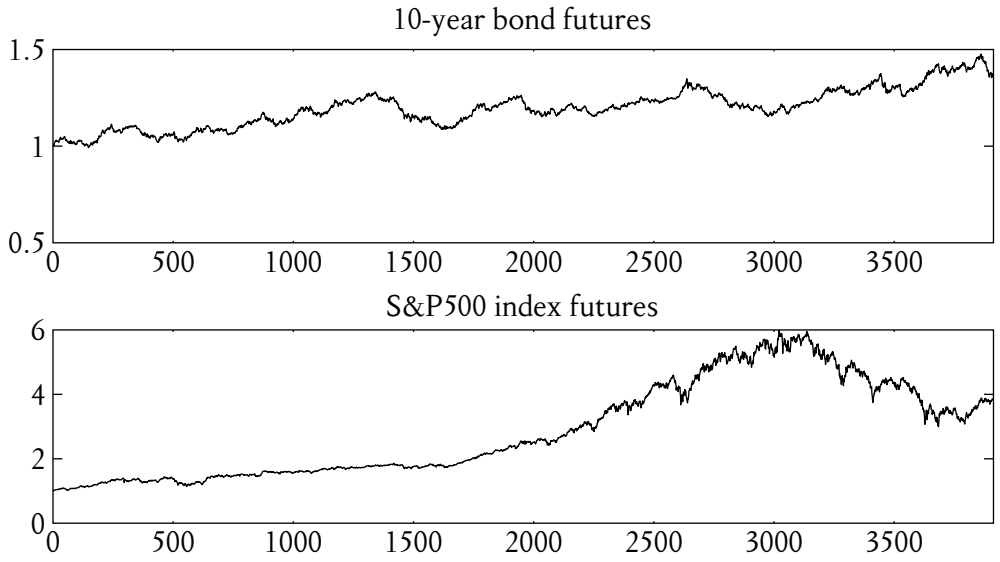


FIGURE A.3 – S&P500 index futures and 10-year bond futures.

correlation matrix :

$$R_1 = \begin{bmatrix} 1 & \\ 0,5189 & 1 \end{bmatrix} \qquad R_2 = \begin{bmatrix} 1 & \\ 0.2924 & 1 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & \\ -0.1853 & 1 \end{bmatrix} \qquad R_4 = \begin{bmatrix} 1 & \\ -0.1597 & 1 \end{bmatrix}$$

transition probabilities

$$A^1 = \begin{bmatrix} 0.9797 & 0.9365 \\ 0.0203 & 0.0635 \end{bmatrix}$$

$$A_1^2 = \begin{bmatrix} 0.9919 & 1.4e^{-4} \\ 0.0055 & 0.7876 \end{bmatrix} \quad A_1^2 = \begin{bmatrix} 0.8139 & 0.0618 \\ 0.1853 & 0.9299 \end{bmatrix}$$

$$e_1^2 = 5.85e-4,\ e_2^2 = 0.2123,\ e_3^2 = 7.7e-4,\ e_4^2 = 0.0084$$

$$\pi_1^2 = 0.0067,\ \pi_2^2 = 0.9933,\ \pi_3^2 = 0.9933,\ \pi_4^2 = 0.0067$$

TABLE A.1 – Estimated parameters for the correlations of the second real data application.

correlation matrix :

$$R_1 = \begin{bmatrix} 1 & & & \\ -0,4860 & 1 & & \\ -0,3943 & 0,8125 & 1 & \\ -0,5004 & 0,9488 & 0,8032 & 1 \end{bmatrix} \quad R_2 = \begin{bmatrix} 1 & & & \\ -4e-5 & 1 & & \\ -3e-5 & 8e-05 & 1 & \\ -5e-5 & 9e-5 & 8e-5 & 1 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & & & \\ -0,8746 & 1 & & \\ -0,6912 & 0,7720 & 1 & \\ -0,8299 & 0,9214 & 0,7587 & 1 \end{bmatrix} \quad R_4 = \begin{bmatrix} 1 & & & \\ -0,7337 & 1 & & \\ -0,5798 & 0,6476 & 1 & \\ -0,6962 & 0,7730 & 0,6365 & 1 \end{bmatrix}$$

transition probabilities

$$A^1 = \begin{bmatrix} 0,9933 & 0,4167 \\ 0,0066 & 0,5832 \end{bmatrix}$$

$$A_1^2 = \begin{bmatrix} 0,6711 & 0,0084 \\ 0,0115 & 0,9915 \end{bmatrix} \quad A_1^2 = \begin{bmatrix} 0,9835 & 0,0002 \\ 0,0157 & 0,9571 \end{bmatrix}$$

$$e_1^2 = 0.3172,\ e_2^2 = 8.21e-05,\ e_3^2 = 7.37e-04,\ e_4^2 = 0.0425$$

$$\pi_1^2 = 0.9933,\ \pi_2^2 = 0.0066,\ \pi_3^2 = 0.9933,\ \pi_4^2 = 0.0067$$

TABLE A.2 – Estimated parameters for the correlations of the second real data application.

ANNEXE

B

B.1 Stability of stochastic matrix under kronecker product

A matrix $A = \big((a_{ij})\big)_{1 \leqslant i,j \leqslant n}$ of $M_n(\mathbb{R})$ with a_{ij} the i^{th} row and j^{th} column element is said stochastic if $\forall (i,j) \in [0,1]^2$:

$$a_{ij} \in (0,1) \tag{B.1}$$

and $\forall i \in \{1,...,n\}$:

$$\sum_{i=1}^{n} a_{ij} = 1 \tag{B.2}$$

Let P_n the set of stochastic matrix of order n. To show that P_n is stable under kronecker product, we consider two stochastic matrix $A = \big((a_{ij})$ and $B = \big((b_{ij})$ with $1 \leqslant i,j \leqslant n$. Let C such that :

$$C = A \otimes B \tag{B.3}$$

Then by definition, the kronecker product, denoting by $\otimes$, of A and B is equal to :

$$C = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} & \cdots & a_{11}b_{1n} & a_{12}b_{1n} & \cdots & a_{1n}b_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{11}b_{n1} & \cdots & a_{11}b_{nn} & a_{12}b_{nn} & \cdots & a_{1n}b_{nn} \\ a_{21}b_{11} & \cdots & a_{21}b_{1n} & a_{22}b_{1n} & \cdots & a_{2n}b_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1}b_{n1} & \cdots & a_{n1}b_{nn} & a_{n2}b_{nn} & \cdots & a_{nn}b_{nn} \end{bmatrix} \tag{B.4}$$

which is a matrix of size $n^n \times n^n$. Since a_{ij} and b_{ij} are in $[0,1]$, we have for each $0 \leqslant k,l \leqslant n^n$:

$$0 \leqslant c_{kl} \leqslant 1 \tag{B.5}$$

If we calculate the sum of the first row, we have :

$$\sum_{k=1}^{n^n} c_{k1} = \sum_{j=1}^{n} \big(a_{j1} \sum_{i=1}^{n} b_{i1}\big) \tag{B.6}$$

$$= \sum_{j=1}^{n} a_{j1} \times 1 \tag{B.7}$$

$$= 1 \tag{B.8}$$

and similarly for the following columns. Thus C is indeed a stochastic matrix. This intersting result allows to have a representation of any factorial hidden Markov model with M parallel chains with N states into a regular hidden Markov model with a transition matrix such $\Upsilon = \bigotimes_{i=1}^{M} \mathrm{P}^{\mathrm{i}}$.

B.2 Smoothed probabilities

Apply Kim's filter to the regular representation Markov-switching representation is an easy way to compute smoothed probabilities. But one drawback is they refer to a combination of state since the transition matrix is the kronecker product of the transition matrix of the parallel chains. One element of the resulting smoothing probabilities vector can then de decomposed as :

$$\hat{\xi}_{t|T} = \mathbb{P}[\upsilon_t = j | \mathscr{F}_T, \theta] \tag{B.9}$$

$$= \mathbb{P}[s_t^1 = j | \mathscr{F}_T, \theta] \times ... \times \mathbb{P}[s_t^M = j | \mathscr{F}_T, \theta] \tag{B.10}$$

Making a correspondance between the smoothed probabilities of independant chains and regular representation is simple using basic linear algebra. Regimes of each chains are represented by an identity matrix $\mathbf{I}_N$ of size N, where each diagonal element represents a regime. To make the computation of the smoothed probabilities of the i^{th} regime of the chain m, this chains is represented by a matrix of zeros except the except the diagonal element of the i^{th} regime whose value is one. Then, the resulting smoothing probabilites are computed as :

$$\mathbb{P}[s_t^1 = j|\mathscr{F}_T, \theta] = \mathbf{Tr}\Big[\big[[\mathbf{I}_N^1 \otimes ... \otimes \mathbf{I}_N^m \otimes ... \otimes \mathbf{I}_N^M]\hat{\xi}_{t|T}\big]1_{M^N}\Big] \tag{B.11}$$

where $\mathbf{Tr}[\cdot]$ is the trace operator and 1_{M^N} a vector of ones with length M^N.
As an exemple, we consider a three chains FHM containing two regimes each. The transition matrix for each chain are expressed as :

$$\mathrm{P^a} = \begin{bmatrix} a11 & a21 \\ a12 & a22 \end{bmatrix}, \mathrm{P^b} \begin{bmatrix} b11 & b21 \\ b12 & b22 \end{bmatrix}, \mathrm{P^c} = \begin{bmatrix} c11 & c21 \\ c12 & c22 \end{bmatrix} \tag{B.12}$$

The transition matrix of the regular Markov-switching formulation is then of size $2^3 \times 2^3$ and is expressed as :

$$\Upsilon = \mathrm{P^a} \otimes \mathrm{P^b} \otimes \mathrm{P^c} \tag{B.13}$$

which gives :

$$\Upsilon = \begin{bmatrix} a11b11c11 & a11b11c21 & a11b21c11 & a11b21c21 & a21b11c11 & a21b11c21 & a21b21c11 & a21b21c21 \\ a11b11c12 & a11b11c22 & a11b21c12 & a11b21c22 & a21b11c12 & a21b11c22 & a21b21c12 & a21b21c22 \\ a11b12c11 & a11b12c21 & a11b22c11 & a11b22c21 & a21b12c11 & a21b12c21 & a21b22c11 & a21b22c21 \\ a11b12c12 & a11b12c22 & a11b22c12 & a11b22c22 & a21b12c12 & a21b12c22 & a21b22c12 & a21b22c22 \\ a12b11c11 & a12b11c21 & a12b21c11 & a12b21c21 & a22b11c11 & a22b11c21 & a22b21c11 & a22b21c21 \\ a12b11c12 & a12b11c22 & a12b21c12 & a12b21c22 & a22b11c12 & a22b11c22 & a22b21c12 & a22b21c22 \\ a12b12c11 & a12b12c21 & a12b22c11 & a12b22c21 & a22b12c11 & a22b12c21 & a22b22c11 & a22b22c21 \\ a12b12c12 & a12b12c22 & a12b22c12 & a12b22c22 & a22b12c12 & a22b12c22 & a22b22c12 & a22b22c22 \end{bmatrix}$$

This a transition matrix with eight regimes. Thus the resulting smoothed probabilities using Kim's filter, defining by equation (3.33), can be decomposed as :

$$\begin{bmatrix} \mathbb{P}[\upsilon_t = 1|\mathscr{F}_T, \theta] \\ \mathbb{P}[\upsilon_t = 2|\mathscr{F}_T, \theta] \\ \mathbb{P}[\upsilon_t = 3|\mathscr{F}_T, \theta] \\ \mathbb{P}[\upsilon_t = 4|\mathscr{F}_T, \theta] \\ \mathbb{P}[\upsilon_t = 5|\mathscr{F}_T, \theta] \\ \mathbb{P}[\upsilon_t = 6|\mathscr{F}_T, \theta] \\ \mathbb{P}[\upsilon_t = 7|\mathscr{F}_T, \theta] \\ \mathbb{P}[\upsilon_t = 8|\mathscr{F}_T, \theta] \end{bmatrix} = \begin{bmatrix} \mathbb{P}[s_t^a = 1, s_t^b = 1, s_t^c = 1|\mathscr{F}_T, \theta] \\ \mathbb{P}[s_t^a = 1, s_t^b = 1, s_t^c = 2|\mathscr{F}_T, \theta] \\ \mathbb{P}[s_t^a = 1, s_t^b = 2, s_t^c = 1|\mathscr{F}_T, \theta] \\ \mathbb{P}[s_t^a = 1, s_t^b = 2, s_t^c = 2|\mathscr{F}_T, \theta] \\ \mathbb{P}[s_t^a = 2, s_t^b = 1, s_t^c = 1|\mathscr{F}_T, \theta] \\ \mathbb{P}[s_t^a = 2, s_t^b = 1, s_t^c = 2|\mathscr{F}_T, \theta] \\ \mathbb{P}[s_t^a = 2, s_t^b = 2, s_t^c = 1|\mathscr{F}_T, \theta] \\ \mathbb{P}[s_t^a = 2, s_t^b = 2, s_t^c = 2|\mathscr{F}_T, \theta] \end{bmatrix} \tag{B.14}$$

where $\mathbb{P}[\upsilon_t = i|\mathscr{F}_T, \theta]$, $i = 1, ..., 8$, denotes the smoothed probability of the of thje regular MS representation. To have the smoothed probability of the second regime of the second chain, we first write the identy matrix for each chains using the rule explained in the last paragraph :

$$\mathbf{I}_2^a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{I}_2^b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{I}_2^c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{B.15}$$

Using equation (B.11), we then have :

$$\mathbb{P}[s_t^b = 2|\mathscr{F}_T, \theta] = \mathbf{Tr}\Big[\big[[\mathbf{I}_2^a \otimes \mathbf{I}_2^b \otimes \mathbf{I}_2^c]\hat{\xi}_{t|T}\big]1_8\Big] \tag{B.16}$$

With :

$$\mathbf{I}_2^a \otimes \mathbf{I}_2^b \otimes \mathbf{I}_2^c = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{B.17}$$

it follows that :

$$\begin{aligned} \mathbb{P}[s_t^b = 2|\mathscr{F}_T, \theta] = \mathbb{P}[s_t^a = 1, s_t^b = 2, s_t^c = 1|\mathscr{F}_T, \theta] + \mathbb{P}[s_t^a = 1, s_t^b = 2, s_t^c = 2|\mathscr{F}_T, \theta] \\ + \mathbb{P}[s_t^a = 2, s_t^b = 2, s_t^c = 1|\mathscr{F}_T, \theta] + \mathbb{P}[s_t^a = 2, s_t^b = 2, s_t^c = 2|\mathscr{F}_T, \theta] \end{aligned} \tag{B.18}$$

B.3 Constant matrix construction

We first define the matrix of the two extrem cases, i.e. when all covariances are in regime 1 :

$$\overline{Q}_{s_t^1=1,s_t^2=1,s_t^3=1} = \begin{bmatrix} q_{11,s_t^1=1} & q_{12,s_t^1=1,s_t^2=1} & q_{13,s_t^1=1,s_t^3=1} \\ & q_{22,s_t^2=1} & q_{23,s_t^2=1,s_t^3=1} \\ & & q_{33,s_t^3=1} \end{bmatrix} \tag{B.19}$$

and the intercept corresponding at the case where all the covariances are in regime 2 :

$$\overline{Q}_{s_t^1=2,s_t^2=2,s_t^3=2} = \begin{bmatrix} q_{11,s_t^1=2} & q_{12,s_t^1=2,s_t^2=2} & q_{13,s_t^1=2,s_t^3=2} \\ & q_{22,s_t^2=2} & q_{23,s_t^2=2,s_t^3=2} \\ & & q_{33,s_t^3=2} \end{bmatrix} \tag{B.20}$$

In the next matrix, non-boxed elements are coming from the constants of the two extrem cases while boxed elements corresponds to the additional elements needed to define the intercept in that case. The six others intercepts can be expressed as :

$$\overline{Q}_{s_t^1=1,s_t^2=1,s_t^3=2} = \begin{bmatrix} q_{11,s_t^1=1} & q_{12,s_t^1=2,s_t^2=1} & \boxed{q_{13,s_t^1=2,s_t^3=2}} \\ & q_{22,s_t^2=1} & \boxed{q_{23,s_t^2=2,s_t^3=2}} \\ & & q_{33,s_t^3=2} \end{bmatrix} \tag{B.21}$$

$$\overline{Q}_{s_t^1=1,s_t^2=2,s_t^3=1} = \begin{bmatrix} q_{11,s_t^1=1} & \boxed{q_{12,s_t^1=1,s_t^2=2}} & q_{13,s_t^1=1,s_t^3=1} \\ & q_{22,s_t^2=2} & \boxed{q_{23,s_t^2=2,s_t^3=1}} \\ & & q_{33,s_t^3=1} \end{bmatrix} \tag{B.22}$$

$$\overline{Q}_{s_t^1=1,s_t^2=2,s_t^3=2} = \begin{bmatrix} q_{11,s_t^1=1} & \boxed{q_{12,s_t^1=2,s_t^2=1}} & \boxed{q_{13,s_t^1=1,s_t^3=2}} \\ & q_{22,s_t^2=2} & q_{23,s_t^2=2,s_t^3=2} \\ & & q_{33,s_t^3=2} \end{bmatrix} \tag{B.23}$$

$$\overline{Q}_{s_t^1=2,s_t^2=1,s_t^3=1} = \begin{bmatrix} q_{11,s_t^1=2} & \boxed{q_{12,s_t^1=2,s_t^2=1}} & \boxed{q_{13,s_t^1=2,s_t^3=1}} \\ & q_{22,s_t^2=1} & q_{23,s_t^2=1,s_t^3=1} \\ & & q_{33,s_t^3=1} \end{bmatrix} \tag{B.24}$$

$$\overline{Q}_{s_t^1=2,s_t^2=1,s_t^3=2} = \begin{bmatrix} q_{11,s_t^1=2} & \boxed{q_{12,s_t^1=2,s_t^2=1}} & q_{13,s_t^1=2,s_t^3=2} \\ & q_{22,s_t^2=1} & \boxed{q_{23,s_t^2=1,s_t^3=2}} \\ & & q_{33,s_t^3=2} \end{bmatrix} \tag{B.25}$$

$$\overline{Q}_{s_t^1=2,s_t^2=2,s_t^3=1} = \begin{bmatrix} q_{11,s_t^1=2} & q_{12,s_t^1=2,s_t^2=2} & \boxed{q_{13,s_t^2=1,s_t^3=1}} \\ & q_{22,s_t^2=2} & \boxed{q_{23,s_t^2=2,s_t^3=1}} \\ & & q_{33,s_t^3=1} \end{bmatrix} \tag{B.26}$$

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